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Lie symmetries of a generalised non-linear Schrödinger equation: I. The symmetry group and its subgroups

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Abstract. The symmetry group of the generalised non-linear Schrödinger equation $i\psi_t + \Delta\psi = a_0\psi + a_1|\psi|^2\psi + a_2|\psi|^4\psi$ in three space dimensions is shown to be the extended Galilei group $\tilde{G}(3)$, for $a_1 a_2 \neq 0$, and the Galilei-similitude group $\tilde{G}^d(3)$ (including a dilation) for $a_1 = 0$ or $a_2 = 0$. All Lie subgroups of $\tilde{G}(3)$ and $\tilde{G}^d(3)$ are found. They will be used in a subsequent paper to obtain group invariant solutions of the equation.

1. Introduction

This study is devoted to a group theoretical investigation of a generalised non-linear Schrödinger equation (GNLSE) in 3+1 dimensions, namely

$$\begin{gathered} i\psi_t + \Delta\psi = a_0\psi + a_1|\psi|^2\psi + a_2|\psi|^4\psi \\ \psi \equiv \psi(x, y, z, t) \in \mathbb{C} \qquad a_i \in \mathbb{R} \qquad i = 1, 2, 3 \qquad (a_1, a_2) \neq (0, 0) \end{gathered} \tag{1.1}$$

where Δ is the three-dimensional Laplace operator in Euclidean 3-space and a_i are constants. This type of non-linear partial differential equation arises in many physical applications, where it describes wave propagation in non-linear and dispersive media. For instance, it can be obtained in non-linear optics [1] when the wavenumber $\boldsymbol{k}$ of an electromagnetic wave is expanded in a power series in terms of the electric field $E(\boldsymbol{r}; t) = \psi(\boldsymbol{r}, t)\exp(i\boldsymbol{k}\cdot\boldsymbol{r})$. Similar equations occur in the description of the electromagnetic heating of a plasma [2, 3], or in the propagation of water waves in certain regimes. Other applications concern the Landau-Ginzburg theory of phase transitions [4], or studies of various biological systems [5].

Previous studies of equation (1.1) were, to our knowledge, restricted to the one-dimensional case (i.e. $\Delta = \partial^2/\partial_x^2$). The equation was shown to have solitary wave solutions [6]. Numerical studies indicate that these are not solitons, i.e. that two solitary waves of (1.1) interact inelastically [7].

The GNLSE does not belong to the class of integrable non-linear evolution equations [8, 9] even in 1+1 dimensions, still less in 3+1. Thus, no Lax pair exists and no linear techniques are available for solving this equation. Exact solitons and multisolitons are hence not to be expected.

Our aim is to apply the techniques of Lie group theory to this equation in order to obtain particular exact solutions and to study their properties. The method consists of several steps.

(i) Find the Lie group G of point transformations

$$\tilde{\boldsymbol{x}} = \Lambda_g(\boldsymbol{x}, \psi) \qquad \tilde{\psi} = \Omega_g(\boldsymbol{x}, \psi) \tag{1.2}$$

leaving the equation invariant. In other words, the transformations (1.2) are such that $\tilde{\psi}(\tilde{x})$ is a solution, whenever $\psi(x)$ is one.

(ii) Find all subgroups of G having orbits of codimension k $(1 \leq k \leq 3)$ in the space of independent variables $x = (x, y, z, t)$.

(iii) Find the invariants of the above subgroups in the space of dependent and independent variables and express the dependent variables in terms of them. In the case under consideration this provides expressions of the type

$$\psi(x, y, z, t) = \alpha(x, y, z, t)\phi(\xi_1, \ldots, \xi_k) \tag{1.3}$$

where α and ξ_i $(i = 1, \ldots, k \leq 3)$ are known functions of the independent variables x, y, z and t.

(iv) Substitute (1.3) into the original equation (1.1) and obtain a differential equation in k variables for the function ϕ. Since we have $1 \leq k \leq 3$ a 'symmetry reduction' is achieved.

(v) Solve the reduced equation for $\phi(\xi_1, \ldots, \xi_k)$ and substitute back into (1.3) to obtain solutions of the original equation. The obtained solutions will be invariant under the considered subgroup of G.

The method described above is a standard one, going back to Lie [10] and is described in various contemporary books [11-14]. A new aspect is that the algorithm for finding the Lie group of point transformations leaving a system of differential equations invariant has been computerised (using symbolic languages, such as MACSYMA [15] or REDUCE [16]). Furthermore, methods have been developed for classifying subalgebras of Lie algebras into conjugacy classes under the action of some group of automorphisms, in particular the group of inner automorphisms [17-22]. Each conjugacy class of subgroups of G, under the action of G itself, provides a different type of group invariant solution and in particular, a different reduced equation.

We shall concentrate in this paper on subgroups with generic orbits of codimension $k = 1$ in spacetime (x, y, z, t). For these subgroups $\phi(\xi)$ in (1.3) depends on one variable only and hence satisfies an ordinary differential equation. While there is no guarantee that we will be able to solve this equation analytically for all reductions, in many cases we can. The solvable cases can again be identified algorithmically. Thus, the obtained ODE may have the Painlevé property (i.e. their solutions have no singularities, other than poles, depending on the initial conditions) [23-25]. Such equations can be integrated in terms of known transcendents [23] or their generalisations [26]. A MACSYMA program has been written to help identify equations with the Painlevé property [25]. Moreover, the ODE itself may have a non-trivial symmetry group, which makes it possible to reduce the order of the ODE or even reduce it to quadratures.

The method of symmetry reduction has recently been applied in a systematic manner to relativistically invariant equations [27], in particular to the field equations of classical relativistic ϕ^6 field theories [28, 29]. This provided a large number of new exact solutions. The method has also been applied to the Kadomtsev-Petviashvili equation [30, 31], the Davey-Stewartson equations [32], the three-wave equations [33] and other completely integrable equations [34] in more than 1+1 dimensions. There the symmetry groups of point transformations turn out to be infinite dimensional and to have a very specific loop-group structure.

This paper is devoted to group theoretical preliminaries. In § 2 we establish that the GNLSE (1.1) is, for $a_1 a_2 \neq 0$, invariant under the extended Galilei group $\tilde{G}$. For $a_1 = 0$ (and also for $a_1 \neq 0$, $a_2 = 0$), independently of a_0, it is invariant under a larger group, namely the extended Galilei group, further extended by a dilation. We shall call this group the Galilei-similitude group and denote it $\tilde{G}^d$. In § 3 we present a classification of subalgebras of the corresponding Lie algebras $\tilde{g}$ and $\tilde{g}^d$ In a subsequent paper we shall single out all classes of Lie subgroups of $\tilde{G}$ and $\tilde{G}^d$, having generic orbits of codimension 1. Each one of them will be used to reduce the GNLSE to an ODE which will then be further analysed.

2. Symmetry group of the equation

In order to find the symmetry group of equation (1.1) we apply an algebraic approach [11]. We look for an algebra of vector fields of the form

$$V = \eta_1\partial_x + \eta_2\partial_y + \eta_3\partial_z + \eta_4\partial_t + \phi_1\partial_{u_1} + \phi_2\partial_{u_2} \tag{2.1}$$

where η_i and ϕ_a are functions of x, y, z, t, u_1, u_2 and where u_1, u_2 are the real and imaginary parts of the solution

$$\psi(x, y, z, t) = u_1(x, y, z, t) + \mathrm{i}u_2(x, y, z, t) \qquad u_1, u_2 \in \mathbb{R}. \tag{2.2}$$

The coefficients η_i and ϕ_a $(i = 1, \ldots, 4; a = 1, 2)$ in (2.1) are determined from the requirement that the second prolongation of V should annihilate the equation on the solution set of the equation. This was implemented using a MACSYMA program [15] that provided a set of 41 determining equations; they are quite easy to solve.

The results can be summarised as follows.

(1) For $a_1 \neq 0$, $a_2 \neq 0$, equation (1.1) is invariant only under the extended Galilei group $\tilde{G} \equiv \tilde{G}(3)$. A convenient basis for its Lie algebra is provided by three translations p_i, three rotations j_i, three proper Galilei transformations k_i, one time translation t and one change of phase generator m. We have

$$\begin{aligned}
&t = \partial_t + a_0(u_2\partial_{u_1} - u_1\partial_{u_2}) \\
&p_1 = \partial_x \qquad p_2 = \partial_y \qquad p_3 = \partial_z \\
&j_1 = z\partial_y - y\partial_z \qquad j_2 = x\partial_z - z\partial_x \qquad j_3 = y\partial_x - x\partial_y \\
&k_1 = t\partial_x - \tfrac{1}{2}x(u_2\partial_{u_1} - u_1\partial_{u_2}) \qquad k_2 = t\partial_y - \tfrac{1}{2}y(u_2\partial_{u_1} - u_1\partial_{u_2}) \\
&k_3 = t\partial_z - \tfrac{1}{2}z(u_2\partial_{u_1} - u_1\partial_{u_2}) \qquad m = u_2\partial_{u_1} - u_1\partial_{u_2}.
\end{aligned} \tag{2.3}$$

(2) For $a_1 = 0$, $a_2 \neq 0$ or $a_1 \neq 0$, $a_2 = 0$ the invariance group is somewhat larger; namely we obtain the extended Galilei group, further extended by a dilation. We shall, by analogy with the relativistic case, call this group the extended Galilei-similitude group $GS(3) \equiv \tilde{G}^d$ (in three space and one time dimensions). A basis for its Lie algebra consists of the 11 operators (2.3) and the dilation generator

$$d = 2t\partial_t + (x\partial_x + y\partial_y + z\partial_z) - \delta(u_1\partial_{u_1} + u_2\partial_{u_2}) + 2a_0t(u_2\partial_{u_1} - u_1\partial_{u_2}) \tag{2.4}$$

$$\delta = \begin{cases} \tfrac{1}{2} & \text{for } a_2 \neq 0 \\ 1 & \text{for } a_1 \neq 0. \end{cases}$$

The group transformation can easily be obtained from (2.3) and (2.4), namely

$$\begin{aligned}
&\tilde{x}_i = \mathrm{e}^{\lambda/2}[R_{ik}x_k - x_{i0} + v_i(t - t_0)] \\
&\tilde{t} = \mathrm{e}^{\lambda}(t - t_0) \\
&\tilde{\psi} = e^{-\lambda\delta/2}\psi \exp \tfrac{1}{2}\mathrm{i}[v_i(R_{ik}x_k - x_{i0}) + \tfrac{1}{2}v^2(t - t_0) + \alpha + 2a_0(t - t_0)(1 - \mathrm{e}^{\lambda})]
\end{aligned} \tag{2.5}$$

where we have put $x_1 \equiv x$, $x_2 = y$, $x_3 = z$. The parameters x_{i0}, t_0, v_i, α and λ correspond to space translations, time translations, Galilei boosts, change of phase and dilations, respectively. The orthogonal matrix R_{ik} ($RR^{\mathrm{T}} = I_3$) corresponds to rotations.

This means that if $\psi(x, y, z, t)$ is a solution of (1.1), then so is

$$\tilde{\psi}(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) = \mathrm{e}^{-\lambda\delta/2}\, \psi\,(R_{ik}\,(\mathrm{e}^{-\lambda/2}\tilde{x}_i - v_i\tilde{t}\,\mathrm{e}^{-\lambda} + x_{i0}), \mathrm{e}^{-\lambda}\,\tilde{t} + t_0)$$

$$\times \exp \tfrac{1}{2}\mathrm{i}\,[v_i\,(\mathrm{e}^{-\lambda/2}\tilde{x}_i - v_i\tilde{t}\,\mathrm{e}^{-\lambda}) + \tfrac{1}{2}v^2\,\mathrm{e}^{-\lambda}\,\tilde{t} + \alpha + 2a_0\tilde{t}(\mathrm{e}^{-\lambda} - 1)]. \tag{2.6}$$

Discrete transformations leaving equation (1.1) invariant are:

(i) reflections in the coordinate planes P_i

$$x_i \to -x_i \qquad t \to t \qquad \psi \to \psi \qquad i = 1, 2 \text{ or } 3 \tag{2.7}$$

(ii) time reversal

$$T\colon \boldsymbol{r} \to \boldsymbol{r},\ t \to -t,\ \psi \to \psi^*. \tag{2.8}$$

Note that the parity operator is $P = P_1P_2P_3$.

We shall denote the extended Galilei algebra with basis (2.3) by $\tilde{\mathrm{g}}$, and the extended Galilei-similitude algebra with basis (2.3) and (2.4) by $\tilde{\mathrm{g}}^{\mathrm{d}}$. Both of these Lie algebras, as well as the corresponding Lie groups, are of considerable interest in physics. We shall present complete subalgebra classifications in both cases, going beyond the low-dimensional subalgebras needed in the present context. Subalgebras of the algebra $\tilde{\mathrm{g}}$ have been studied by Sorba [35]. We go well beyond his results in identifying the isomorphy classes of subalgebras of $\tilde{\mathrm{g}}$ and their properties. Moreover, we present a classification under the group $\tilde{\mathrm{G}}$ and $\tilde{\mathrm{G}}^{\mathrm{d}}$. The subalgebras of $\tilde{\mathrm{g}}^{\mathrm{d}}$ are studied here for the first time.

Like all finite-dimensional Lie algebras, $\tilde{\mathrm{g}}$ and $\tilde{\mathrm{g}}^{\mathrm{d}}$ allow Levi decompositions [36], $\mathrm{g} \sim \mathrm{S} \mathbin{\supset\!\!\!\!+} \mathrm{R}$, where S is semisimple and R is the radical (maximal solvable ideal). We have

$$\begin{aligned} &\tilde{\mathrm{g}} \sim \{j_1, j_2, j_3\} \mathbin{\supset\!\!\!\!+} \{t, k_1, k_2, k_3, p_1, p_2, p_3, m\} \\ &\tilde{\mathrm{g}}^{\mathrm{d}} \sim \{j_1, j_2, j_3\} \mathbin{\supset\!\!\!\!+} \{d, t, k_1, k_2, k_3, p_1, p_2, p_3, m\} \end{aligned} \tag{2.9}$$

i.e. $\mathrm{S} = \{j_1, j_2, j_3\} \sim \mathrm{o}(3)$.

The o(3) algebra $\{j_1, j_2, j_3\}$ constitutes the semisimple component of each of these algebras. The remaining infinitesimal operators span the radicals. The radical of $\tilde{\mathrm{g}}$ is actually nilpotent and contains the Heisenberg algebra $\{k_1, k_2, k_3, p_1, p_2, p_3, m\} \sim \mathrm{h}(3)$.

We shall, in our subalgebra classification, make use of different decompositions, namely

$$\begin{aligned} &\tilde{\mathrm{g}} \sim \mathrm{f} \mathbin{\supset\!\!\!\!+} \mathrm{n} \qquad \mathrm{f} \sim \{j_1, j_2, j_3, k_1, k_2, k_3\},\ \mathrm{n} \sim \{t, p_1, p_2, p_3, m\} \\ &\tilde{\mathrm{g}}^{\mathrm{d}} \sim \{d\} \mathbin{\supset\!\!\!\!+} \tilde{\mathrm{g}}. \end{aligned} \tag{2.10}$$

In (2.10) n is an Abelian ideal and f is a factor algebra $\mathrm{f} \sim \tilde{\mathrm{g}}/\mathrm{n}$ which is itself a Lie algebra, isomorphic to the Euclidean Lie algebra e(3).

The commutation relations for the two algebras $\tilde{\mathrm{g}}$ and $\tilde{\mathrm{g}}^{\mathrm{d}}$ are given in table 1.

The extended Galilei group $\tilde{\mathrm{G}}$ plays a fundamental role in non-relativistic quantum mechanics. It has been extensively studied, e.g., by Levy-Leblond [37] and Voisin [38]. A large class of equations is invariant under $\tilde{\mathrm{G}}$, in particular any non-linear Schrödinger equation of the form

$$\mathrm{i}\frac{\partial\psi}{\partial t} + \Delta\psi = F(|\psi|)\psi.$$

Table 1. Commutation relations for $\tilde{g}^{d}(3)$ and $\tilde{g}(3)$.

	d	j_1	j_2	j_3	k_1	k_2	k_3	p_1	p_2	p_3	t	m
d	0	0	0	0	k_1	k_2	k_3	$-p_1$	$-p_2$	$-p_3$	$-2t$	0
j_1	0	0	j_3	$-j_2$	0	k_3	$-k_2$	0	p_3	$-p_2$	0	0
j_2	0	$-j_3$	0	j_1	$-k_3$	0	k_1	$-p_3$	0	p_1	0	0
j_3	0	j_2	$-j_1$	0	k_2	$-k_1$	0	p_2	$-p_1$	0	0	0
k_1	$-k_1$	0	k_3	$-k_2$	0	0	0	$\frac{1}{2}m$	0	0	$-p_1$	0
k_2	$-k_2$	$-k_3$	0	k_1	0	0	0	0	$\frac{1}{2}m$	0	$-p_2$	0
k_3	$-k_3$	k_2	$-k_1$	0	0	0	0	0	0	$\frac{1}{2}m$	$-p_3$	0
p_1	p_1	0	p_3	$-p_2$	$-\frac{1}{2}m$	0	0	0	0	0	0	0
p_2	p_2	$-p_3$	0	p_1	0	$-\frac{1}{2}m$	0	0	0	0	0	0
p_3	p_3	p_2	$-p_1$	0	0	0	$-\frac{1}{2}m$	0	0	0	0	0
t	$2t$	0	0	0	p_1	p_2	p_3	0	0	0	0	0
m	0	0	0	0	0	0	0	0	0	0	0	0

Similarly, the group $\tilde{G}^{d}$ is pertinent in the study of scaling phenomena in any non-relativistic quantum theory.

In the classical limit, $\hbar \to 0$, the extended Galilei algebra $\tilde{g}$ and the extended Galilei-similitude algebra $\tilde{g}^{d}$ contract to a direct sum of the corresponding non-extended algebras g and g^{d} with a one-dimensional algebra $\{m\}$. Their subalgebras and the corresponding subgroups of the non-extended groups G and G^{d} will be of use, e.g., in the study of classical non-relativistic integrable systems. A classification of the subalgebras of g and g^{d} is presented elsewhere [39].

3. The subalgebra classification

We shall classify the subalgebras of the extended Galilei algebra $\tilde{g}$ into conjugacy classes under the action of the connected component of the extended Galilei group $\tilde{G}_0 \equiv G_0(3)$, under the group $\tilde{G} \equiv \tilde{G}(3)$ that includes parity P and time reversal T and also under the connected component of the extended Galilei-similitude group $\tilde{G}_0^{d}$ and under $\tilde{G}^{d} = GS(3)$, including P and T. The subalgebras of the extended Galilei-similitude algebra $\tilde{g}^{d}$ are classified into conjugacy classes under the action of $GS_0(3)$ and GS(3).

The method to be used for $\tilde{g}$ was developed [17] in connection with a classification of subalgebras of the Poincaré algebra p(3, 1). The one for $\tilde{g}^{d}$ was first presented in connection with the relativistic similitude algebra sim(3, 1) [18]. They have been applied to find all closed connected subgroups of such fundamental groups of physics as the Poincaré [17] and similitude groups [18], the two de Sitter groups [19, 20], the optical group [22] and the Schrödinger group in 1+1 and 2+1 dimensions [21, 22].

3.1. Subalgebras of the extended Galilei algebra $\tilde{g}$

We make use of the semidirect sum decomposition (2.10). The classification procedure can be formulated as an algorithm, consisting of several steps. Consider a Lie algebra

$$\mathrm{l} = \mathrm{f} \mathbin{\supset\!\!\!\!+} \mathrm{n} \qquad (\mathrm{n} = \text{ideal in } \mathrm{l}).$$

(1) Classify all subalgebras of the algebra f into conjugacy classes under the action of the group F = expf. Choose a representative f_i of each conjugacy class of

subalgebras in such a manner that the normaliser $\mathrm{nor}_{\mathrm{f}}\, f_i$ of f_i in f is also in the list. We recall here the definition of the normaliser of a subalgebra in a Lie algebra:

$$\mathrm{nor}_{\mathrm{f}}\, f_i = \{\mathrm{x} \in \mathrm{f} \mid [x, f_i] \subseteq f_i\}. \tag{3.1}$$

Obviously we have

$$f_i \subseteq \mathrm{nor}_{\mathrm{f}}\, f_i \subseteq \mathrm{f}.$$

The trivial subalgebras $f_0 = \mathrm{f}$ and $f_p = \{\varnothing\}$ must be included in the list.

In the case under consideration we have $\mathrm{f} \sim \mathrm{e}(3)$. The subalgebras of the Euclidean algebra e(3) have already been classified [19] and we reproduce the results in table 2. For future use we give the normaliser of each subalgebra f_i both in f and in $\tilde{\mathrm{g}}$ (see column 5).

Table 2. Representatives of conjugacy classes of subalgebras of the Euclidean algebra $\mathrm{f} \sim \mathrm{e}(3)$.

					Range of parameters			
Dimension	Notation	Basis	$\mathrm{nor}_{\mathrm{f}}\, f_i$	$\mathrm{nor}_{\tilde{\mathrm{g}}}\, f_i$	$\tilde{\mathrm{G}}_0$	$\tilde{\mathrm{G}}$	$\tilde{\mathrm{G}}_0^{\mathrm{d}}$	$\tilde{\mathrm{G}}^{\mathrm{d}}$
6	f_0	$\boldsymbol{j}, \boldsymbol{k}$	f_0	$\{f_0, m\}$				
4	f_1	$j_3, \boldsymbol{k}$	f_1	$\{f_1, m\}$				
3	f_2	$\boldsymbol{j}$	f_2	$\{f_2, t, m\}$				
3	f_3	j_3, k_1, k_2	f_1	$\{f_1, p_3, m\}$				
3	f_4	$\boldsymbol{k}$	f_0	$\{f_0, m\}$				
3	f_5^a	$j_3 + ak_3, k_1, k_2$	f_1	$\{f_1, m\}$	$a \neq 0$	$a > 0$	$a = \pm 1$	$a = 1$
2	f_6	j_3, k_3	f_6	$\{f_6, m\}$				
2	f_7	k_1, k_2	f_1	$\{f_1, p_3, m\}$				
1	f_8	j_3	f_6	$\{f_1, p_3, t, m\}$				
1	f_9	k_3	f_1	$\{f_1, p_1, p_2, m\}$				
1	f_{10}^a	$j_3 + ak_3$	f_6	$\{f_6, m\}$	$a \neq 0$	$a > 0$	$a = \pm 1$	$a = 1$
0	f_{11}	$\varnothing$	f_0	$\tilde{\mathrm{g}}(3)$				

Every subalgebra of f is conjugated to precisely one algebra in table 2. Which classification group is used only influences the range of parameters introduced in column 3 (see column 6).

(2) For each subalgebra $f_i \subset \mathrm{f}$ find all invariant subspaces $n_{i,\alpha} \subseteq \mathrm{n}$ ($[f_i, n_{i,\alpha}] \subseteq n_{i,\alpha}$) that are also subalgebras of n. Since in the case under consideration the ideal n is Abelian, every subspace of n is a subalgebra. Classify the invariant subspaces for each f_i into conjugacy classes under the action of $\mathrm{Nor}_{\mathrm{G}}\, f_i$, where

$$\mathrm{Nor}_{\mathrm{G}}\, f_i = \{\mathrm{g} \in \mathrm{G} \mid \mathrm{g} f_i \mathrm{g}^{-1} \subseteq f_i\} \tag{3.2}$$

and G is the classifying group under consideration. Choose a representative $n_{i,\alpha}$ of each conjugacy class. A list of representatives of all G conjugacy classes of *splitting* subalgebras of l is obtained by taking the set of all algebras that are the algebraic sums of the spaces f_i and $n_{i,\alpha}$:

$$f_i \dotplus n_{i,\alpha} \qquad \forall i, \forall \alpha. \tag{3.3}$$

(3) Find all *non-splitting* subalgebras of l. These are subalgebras containing, in any basis, at least one basis element not contained in the ideal n, nor in the factor algebra f (not even after conjugation by the classifying group). To obtain all non-splitting subalgebras of a Lie algebra $\mathrm{l} = \mathrm{f} \oplus \mathrm{n}$ we choose a basis for n, say $\{X_1, \ldots, X_p\}$.

We then run through the list of all splitting subalgebras $\{f_i + n_{i,\alpha}\}$. For each of them we have a basis

$$f_i = \{B_1, \ldots, B_r\} \qquad n_{i,\alpha} = \{X_1, \ldots, X_p\}. \tag{3.4}$$

All non-splitting subalgebras of l, related to the splitting subalgebra (3.4), will have bases in the form

$$\begin{aligned} &\left\{B_a + \sum_{j=1}^{m} C_{aj} Y_j, X_i\right\} && a = 1, \ldots, r && i = 1, \ldots, p \\ &Y_j \in \mathrm{n}/n_{i,\alpha} && j = 1, \ldots, m && m + p = \dim \mathrm{n}. \end{aligned} \tag{3.5}$$

The constants $C_{aj} \in \mathbb{R}$ are subject to the condition that (3.5) must be the basis of a Lie algebra. The classification of all non-splitting subalgebras amounts to a classification of all the algebras (3.5) under the action of the group $\mathrm{Nor}_{\mathrm{G}}(f_i \dotplus n_{i,\alpha}) \oplus \mathrm{N}$ and to the choice of a representative of each conjugacy class (where $N = \exp n$).

In cohomological terms, the condition that the elements in (3.5) form the basis of a Lie algebra means that the coefficients C_{aj} form 1-cocycles. Those that can be eliminated by transformations in the invariant subgroup N form 1-coboundaries. If all 1-cocycles are 1-coboundaries then the subalgebra is a splitting one.

Combining together the representative lists of splitting and non-splitting subalgebras of l, we obtain a normalised representative list of all subalgebras of l (the normaliser of every algebra in the list is also in the list).

The results of the classification of the subalgebras of the extended Galilei algebra $\tilde{\mathrm{g}}$ are given in table 3. In column 2 we give some information on the isomorphy class of the subalgebra. For subalgebras of dimensions $d \leq 5$ and nilpotent algebras of dimension $d = 6$ a complete classification of isomorphy classes exists [40–44]. For these algebras we present the isomorphy class in column 2, following the notations of [44]. For $d = 7, \ldots, 11$ and $d = 6$ non-nilpotent, we give whatever information is available in column 2. In particular, if a subalgebra is decomposable, then its indecomposable components are identified. Subalgebras containing the three rotations $j = \{j_1, j_2, j_3\}$ have Levi decompositions in which the semisimple subalgebra is o(3), and all other basis elements span a nilpotent ideal (the radical, which is also the nilradical). Algebras containing one element involving a rotation (j_3, $j_3 + at$, $j_3 + ak_3$, $j_3 + ap_3$, $j_3 + am$, or $j_3 + ak_3 + bt$) are solvable, but not nilpotent. The bases are presented in column 3 in such a way that the nilradical (NR), i.e. the maximal nilpotent ideal, is obtained by simply omitting the basis element involving j_3. Some information on the nilradicals of the solvable subalgebras is also given in column 2. The basis elements to the right of the semicolon in column 3 span the derived algebra of the subalgebra.

The normaliser of each algebra in the extended Galilei algebra $\tilde{\mathrm{g}}$, and extended Galilei-similitude algebra $\tilde{\mathrm{g}}^{\mathrm{d}}$, are presented in columns 4 and 5, respectively. The subalgebras are denoted $\tilde{g}_{i,k}$, where i denotes the dimension and k labels different subalgebra classes of the same dimension. Many subalgebras depend on parameters $a, b, \ldots \in \mathbb{R}$. Their range is indicated in column 6, for conjugacy considered under the proper extended Galilei group $\tilde{\mathrm{G}}_0$, the group $\tilde{\mathrm{G}}$ including parity P and time reversal T, the proper extended Galilei-similitude group $\tilde{\mathrm{G}}_0^{\mathrm{d}}$ and also the group $\tilde{\mathrm{G}}^{\mathrm{d}}$, including P and T.

3.2. Subalgebras of the extended Galilei-similitude algebra $\tilde{g}^d$

We apply a somewhat modified version of the classification procedure used above [18].

Table 3. Representatives of conjugacy classes of subalgebras of the extended Galilei algebra $\tilde{g} = \tilde{g}(3)$. The classification group is specified in column 6 and only influences the range of parameters (if any). These are also representatives of $\tilde{G}_0^d$ or $\tilde{G}^d$ conjugacy classes of subalgebras of $\tilde{g}^d$ not involving dilations. For $\tilde{g}_4'$ and $\tilde{g}_4''$ see table 4.

					Range of parameters			
Number	Isomorphism class and comments	Basis	$\mathrm{nor}_{\tilde{g}}$	$\mathrm{nor}_{\tilde{g}^d}$	$\tilde{G}_0$	$\tilde{G}$	$\tilde{G}_0^d$	$\tilde{G}^d$
$\tilde{g}_{11,1}$	$\tilde{g}(3)$	$t; \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{p}, m$	$\tilde{g}_{11,1}$	$\tilde{g}_{12,1}^d$				
$\tilde{g}_{10,1}$		$; \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{p}, m$	$\tilde{g}_{11,1}$	$\tilde{g}_{12,1}^d$				
$\tilde{g}_{9,1}$		$j_3, t; k_1, k_2, \boldsymbol{p}, m$	$\tilde{g}_{9,1}$	$\tilde{g}_{10,2}^d$				
$\tilde{g}_{8,1}$	$e(3) \oplus 2L(1,1)$	$\{; \boldsymbol{j}, \boldsymbol{p}\} \oplus \{t;\} \oplus \{m;\}$	$\tilde{g}_{8,1}$	$\tilde{g}_{9,2}^d$				
$\tilde{g}_{8,2}$	$\tilde{g}_{7,15} \oplus L(1,1)$	$\{j_3, t; k_1, k_2, p_1, p_2, m\} \oplus \{p_3;\}$	$\tilde{g}_{9,1}$	$\tilde{g}_{10,2}^d$				
$\tilde{g}_{8,3}$	nilpotent	$t, \boldsymbol{k}; \boldsymbol{p}, m$	$\tilde{g}_{11,1}$	$\tilde{g}_{10,2}^d$				
$\tilde{g}_{8,4}$	solv, NR = $L(6,14;1) \oplus L(1,1)$	$j_3 + ak_3, t; k_1, k_2, \boldsymbol{p}, m$	$\tilde{g}_{9,1}$	$\tilde{g}_{9,1}^d$	$a \neq 0$	$a > 0$	$a = \pm 1$	$a = 1$
$\tilde{g}_{8,5}$	solv, NR = $\tilde{g}_{7,11}$	$j_3, k_3 + at, p_3; k_1, k_2, p_1, p_2, m$	$\tilde{g}_{9,1}$	$\tilde{g}_{9,1}^d$	$a > 0$	$a > 0$	$a = 1$	$a = 1$
$\tilde{g}_{8,6}$	solv, NR = $\tilde{g}_{7,11}$	$j_3 + bk_3, k_3 + at; k_1, k_2, \boldsymbol{p}, m$	$\tilde{g}_{9,1}$	$\tilde{g}_{9,1}^d$	$a > 0$ $b \neq 0$	$a > 0$ $b > 0$	$a = 1$ $b \neq 0$	$a = 1$ $b > 0$
$\tilde{g}_{8,7}$	solv, NR = $h(3)$	$j_3, k_3, p_3; p_1, p_2, k_1, k_2, m$	$\tilde{g}_{9,1}$	$\tilde{g}_{10,2}^d$				
$\tilde{g}_{8,8}$	solv, NR = $h(3)$	$j_3 + at, k_3; k_1, k_2, \boldsymbol{p}, m$	$\tilde{g}_{9,1}$	$\tilde{g}_{9,1}^d$	$a > 0$	$a > 0$	$a = 1$	$a = 1$
$\tilde{g}_{7,1}$	$L(3,4;0) \oplus L(4,1)$	$\{j_3; p_1, p_2\} \oplus \{t, k_3; p_3, m\}$	$\tilde{g}_{7,1}$	$\tilde{g}_{8,15}^d$				
$\tilde{g}_{7,2}$	$L(6,14;1) \oplus L(1,1)$	$\{t, k_1, k_2; p_1, p_2, m\} \oplus \{p_3;\}$	$\tilde{g}_{9,1}$	$\tilde{g}_{10,2}^d$				
$\tilde{g}_{7,3}$	$e(3) \oplus L(1,1)$	$\{; \boldsymbol{j}, \boldsymbol{k}\} \oplus \{m;\}$	$\tilde{g}_{7,3}$	$\tilde{g}_{8,9}^d$				
$\tilde{g}_{7,4}$	$e(3) \oplus L(1,1)$	$\{; \boldsymbol{j}, \boldsymbol{p}\} \oplus \{m;\}$	$\tilde{g}_{8,1}$	$\tilde{g}_{8,10}^d$				
$\tilde{g}_{7,5}$	$e(3) \oplus L(1,1)$	$\{; \boldsymbol{j}, \boldsymbol{p}\} \oplus \{t;\}$	$\tilde{g}_{8,1}$	$\tilde{g}_{9,2}^d$				
$\tilde{g}_{7,6}$	$e(3) \oplus L(1,1)$	$\{; \boldsymbol{j}, \boldsymbol{p}\} \oplus \{t + am\}$	$\tilde{g}_{8,1}$	$\tilde{g}_{8,1}^d$	$a \neq 0$	$a \neq 0$	$a = \pm 1$	$a = \pm 1$
$\tilde{g}_{7,7}$	$\tilde{g}_{6,15} \oplus L(1,1)$	$\{j_3; k_1, k_2; p_1, p_2, m\} \oplus \{k_3;\}$	$\tilde{g}_{8,7}$	$\tilde{g}_{9,6}^d$				
$\tilde{g}_{7,8}$	$\sim \tilde{g}_{7,7}$	$\{j_3; k_1, k_2; p_1, p_2, m\} \oplus \{p_3;\}$	$\tilde{g}_{9,1}$	$\tilde{g}_{10,2}^d$				
$\tilde{g}_{7,9}$	$\tilde{g}_{6,18} \oplus L(1,1)$	$\{j_3 + at; k_1, k_2; p_1, p_2, m\} \oplus \{p_3;\}$	$\tilde{g}_{9,1}$	$\tilde{g}_{9,1}^d$	$a > 0$	$a > 0$	$a = 1$	$a = 1$
$\tilde{g}_{7,10}$	nilpotent, $h(3)$	$\boldsymbol{k}, \boldsymbol{p}; m$	$\tilde{g}_{11,1}$	$\tilde{g}_{12,1}^d$				
$\tilde{g}_{7,11}$	nilpotent	$k_3 + at, k_1, k_2, p_3; p_1, p_2, m$	$\tilde{g}_{9,1}$	$\tilde{g}_{9,1}^d$	$a > 0$	$a > 0$	$a = 1$	$a = 1$
$\tilde{g}_{7,12}$	solv, NR = $L(5,4) \oplus L(1,1)$	$j_3 + ak_3, p_3; k_1, k_2, p_1, p_2, m$	$\tilde{g}_{9,1}$	$\tilde{g}_{9,1}^d$	$a \neq 0$	$a > 0$	$a = \pm 1$	$a = 1$
$\tilde{g}_{7,13}$	$\sim \tilde{g}_{7,12}$	$j_3 + ap_3, k_3; k_1 k_2, p_1, p_2, m$	$\tilde{g}_{8,7}$	$\tilde{g}_{8,7}^d$	$a \neq 0$	$a > 0$	$a = \pm 1$	$a = 1$
$\tilde{g}_{7,14}$	solv, NR = $L(5,4) \oplus L(1,1)$	$j_3 + bk_3 + at, p_3; k_1, k_2, p_1, p_2, m$	$\tilde{g}_{9,1}$	$\tilde{g}_{9,1}^d$	$a > 0$ $b \neq 0$	$a > 0$ $b > 0$	$a = 1$ $b \neq 0$	$a = 1$ $b > 0$
$\tilde{g}_{7,15}$	solv, NR = $L(6,14;1)$	$j_3, t; k_1, k_2, p_1, p_2, m$	$\tilde{g}_{8,2}$	$\tilde{g}_{9,5}^d$				
$\tilde{g}_{7,16}$	$\sim \tilde{g}_{7,15}$	$j_3, k_3 + at; k_1, k_2, p_1, p_2, m$	$\tilde{g}_{8,5}$	$\tilde{g}_{8,5}^d$	$a > 0$	$a > 0$	$a = 1$	$a = 1$
$\tilde{g}_{7,17}$	$\sim \tilde{g}_{7,15}$	$j_3 + ap_3, t; k_1, k_2, p_1, p_2, m$	$\tilde{g}_{8,2}$	$\tilde{g}_{8,2}^d$	$a \neq 0$	$a > 0$	$a = \pm 1$	$a = 1$
$\tilde{g}_{7,18}$	$\sim \tilde{g}_{7,15}$	$j_3 + bp_3, k_3 + at; k_1, k_2, p_1, p_2, m$	$\tilde{g}_{8,5}$	$\tilde{g}_{8,5}^d$	$a > 0$ $b \neq 0$	$a > 0$ $b \neq 0$	$a = 1$ $b \neq 0$	$a = 1$ $b \neq 0$
$\tilde{g}_{6,1}$	$L(3,4;0) \oplus 3L(1,1)$	$\{j_3; p_1, p_2\} \oplus \{t;\} \oplus \{p_3;\} \oplus \{m;\}$	$\tilde{g}_{7,1}$	$\tilde{g}_{8,15}^d$				
$\tilde{g}_{6,2}$	$L(3,4;0) \oplus L(3,1)$	$\{j_3; k_1, k_2\} \oplus \{k_3, p_3; m\}$	$\tilde{g}_{6,2}$	$\tilde{g}_{7,28}^d$				
$\tilde{g}_{6,3}$	$L(3,4;0) \oplus L(3,1)$	$\{j_3; p_1, p_2\} \oplus \{k_3, p_3; m\}$	$\tilde{g}_{7,1}$	$\tilde{g}_{8,15}^d$				
$\tilde{g}_{6,4}$	$L(3,4;0) \oplus L(3,1)$	$\{j_3; p_1, p_2\} \oplus \{k_3 + at, p_3; m\}$	$\tilde{g}_{7,1}$	$\tilde{g}_{7,1}^d$	$a > 0$	$a > 0$	$a = 1$	$a = 1$
$\tilde{g}_{6,5}$	$L(3,4{:}\,0\} \oplus L(3,1)$	$\{j_3; k_1 + ap_2, k_2 - ap_1\}$ $\oplus \{k_3, p_3; m\}$	$\tilde{g}_{6,5}$	$\tilde{g}_{6\,5}^d$	$a > 0$	$a > 0$	$a = 1$	$a = 1$
$\tilde{g}_{6,6}$	$L(4,1) \oplus 2L(1,1)$	$\{t, k_3; p_3, m\} \oplus \{p_1;\} \oplus \{p_2;\}$	$\tilde{g}_{9,1}$	$\tilde{g}_{10,2}^d$				
$\tilde{g}_{6,7}$	$L(5,5) \oplus L(1,1)$	$\{k_3 + at, k_2, p_3; p_2, m\} \oplus \{p_1;\}$	$\tilde{g}_{8,3}$	$\tilde{g}_{8,3}^d$	$a > 0$	$a > 0$	$a = 1$	$a = 1$
$\tilde{g}_{6,8}$	$L(5,4) \oplus L(1,1)$	$\{k_1, k_2, p_1, p_2; m\} \oplus \{p_3;\}$	$\tilde{g}_{9,1}$	$\tilde{g}_{10,2}^d$				
$\tilde{g}_{6,9}$	$L(5,4) \oplus L(1,1)$	$\{k_1, k_2, p_1, p_2; m\} \oplus \{k_3;\}$	$\tilde{g}_{8,7}$	$\tilde{g}_{9,6}^d$				
$\tilde{g}_{6,10}$	$L(5,4) \oplus L(1,1)$	$\{k_1 + ap_3, k_2, p_1, p_2; m\}$ $\oplus \{k_3 + ap_1\}$	$\tilde{g}_{7,10}$	$\tilde{g}_{7,10}^d$	$a > 0$	$a > 0$	$a = 1$	$a = 1$
$\tilde{g}_{6,11}$	$L(6,14;1)$	$t, k_1, k_2; p_1, p_2, m$	$\tilde{g}_{8,2}$	$\tilde{g}_{9,5}^d$				
$\tilde{g}_{6,12}$	$L(6,14;1)$	$k_3 + at, k_1, k_2; p_1, p_2, m$	$\tilde{g}_{8,7}$	$\tilde{g}_{8,7}^d$	$a > 0$	$a > 0$	$a = 1$	$a = 1$
$\tilde{g}_{6,13}$	$L(6,14;1)$	$k_3 + at, k_1 + bp_3, k_2; p_1, p_2, m$	$\tilde{g}_{7,11}$	$\tilde{g}_{7,11}^d$	$a > 0$ $b > 0$	$a > 0$ $b > 0$	$a = 1$ $b > 0$	$a = 1$ $b > 0$
$\tilde{g}_{6,14}$	$L(6,14;1)$	$t, k_1 + ap_3, k_2; p_1, p_2, m$	$\tilde{g}_{7,2}$	$\tilde{g}_{7,2}^d$	$a > 0$	$a > 0$	$a = 1$	$a = 1$
$\tilde{g}_{6,15}$	solv, NR = $L(5,4)$	$j_3; k_1, k_2, p_1, p_2, m$	$\tilde{g}_{9,1}$	$\tilde{g}_{10,2}^d$				

Table 3. (continued)

					Range of parameters			
Number	Isomorphism class and comments	Basis	$\mathrm{nor}_{\tilde{g}}$	$\mathrm{nor}_{\tilde{g}^{d}}$	$\tilde{G}_0$	$\tilde{G}$	$\tilde{G}_0^{d}$	$\tilde{G}^{d}$
$\tilde{g}_{6,16}$	$\sim\tilde{g}_{6,15}$	$j_3+ap_3;\ k_1, k_2, p_1, p_2, m$	$\tilde{g}_{9,1}$	$g_{9,1}^{d}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde{g}_{6,17}$	$\sim\tilde{g}_{6,15}$	$j_3+ak_3;\ k_1, k_2, p_1, p_2, m$	$\tilde{g}_{8,7}$	$\tilde{g}_{8,7}^{d}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde{g}_{6,18}$	solv, NR = L(5, 4)	$j_3+at;\ k_1, k_2, p_1, p_2, m$	$\tilde{g}_{8,2}$	$\tilde{g}_{8,2}^{d}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{6,19}$	$\sim\tilde{g}_{6,18}$	$k_3+at+bj_3;\ k_1, k_2, p_1, p_2, m$	$\tilde{g}_{8,5}$	$\tilde{g}_{8,5}^{d}$	$a>0$ $b\neq 0$	$a>0$ $b\neq 0$	$a=1$ $b\neq 0$	$a=1$ $b\neq 0$
$\tilde{g}_{6,20}$	solv, NR = L(3, 1)⊕2L(1, 1)	$j_3+bt, k_3+at, p_3;\ p_1, p_2, m$	$\tilde{g}_{7,1}$	$\tilde{g}_{7,1}^{d}$	$a>0$ $b\neq 0$	$a>0$ $b>0$	$a=1$ $b\neq 0$	$a=1$ $b>0$
$\tilde{g}_{6,21}$	$\sim\tilde{g}_{6,20}$	$j_3+bt, k_3, p_3;\ p_1 p_2, m$	$\tilde{g}_{7,1}$	$\tilde{g}_{7,1}^{d}$	$b>0$	$b>0$	$b=1$	$b=1$
$\tilde{g}_{6,22}$	solv, NR = 5L(1, 1)	$j_3+ak_3, t;\ \boldsymbol{p}, m$	$\tilde{g}_{7,1}$	$\tilde{g}_{7,1}^{d}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde{g}_{6,23}$	e(3)	$;\ \boldsymbol{j}, \boldsymbol{k}$	$\tilde{g}_{7,3}$	$\tilde{g}_{8,9}^{d}$				
$\tilde{g}_{6,24}$	e(3)	$;\ \boldsymbol{j}, \boldsymbol{p}$	$\tilde{g}_{8,1}$	$\tilde{g}_{8,10}^{d}$				
$\tilde{g}_{5,1}$	5L(1, 1)	$\{p_1;\}\oplus\{p_2;\}\oplus\{p_3;\}\oplus\{t;\}\oplus\{m;\}$	$\tilde{g}_{11,1}$	$\tilde{g}_{12,1}^{d}$				
$\tilde{g}_{5,2}$	L(3, 1)⊕2L(1, 1)	$\{k_1+ap_2, k_2-ap_1;\ m\}$ $\oplus\{k_2+ap_1;\}\oplus\{p_3;\}$	$\tilde{g}_{7,10}$	$\tilde{g}_{7,10}^{d}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{5,3}$	L(3, 1)⊕2L(1, 1)	$\{k_3+ap_2, k_2+bp_1+cp_2-ap_3;\ m\}$ $\oplus\{k_1+bp_2;\}$ $\oplus\{k_2+bp_1+cp_2+ap_3\}$	$\tilde{g}_{7,10}$	$\tilde{g}_{7,10}^{d}$	$a>0$ $b\geqslant 0$ $c\in\mathbb{R}$	$a>0$ $b\geqslant 0$ $c\in\mathbb{R}$	$a=1$ $b\geqslant 0$ $c\in\mathbb{R}$	$a=1$ $b\geqslant 0$ $c\in\mathbb{R}$
$\tilde{g}_{5,4}$	L(3, 1)⊕2L(1, 1)	$\{k_3, p_3;\ m\}\oplus\{k_1;\}\oplus\{k_2+ap_2;\}$	$\tilde{g}_{7,10}$	$\tilde{g}_{7,10}^{d}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde{g}_{5,5}$	L(3, 1)⊕2L(1, 1)	$\{k_3, p_3;\ m\}\oplus(k_1;\}\oplus\{k_2;\}$	$\tilde{g}_{8,7}$	$\tilde{g}_{9,2}^{d}$				
$\tilde{g}_{5,6}$	L(3, 1)⊕2L(1, 1)	$\{k_1, p_1;\ m\}\oplus\{k_2;\}\oplus\{p_3;\}$	$\tilde{g}_{7,10}$	$\tilde{g}_{7,10}^{d}$				
$\tilde{g}_{5,7}$	L(3, 1)⊕2L(1, 1)	$\{k_3, p_3;\ m\}\oplus\{p_1;\}\oplus\{p_2;\}$	$\tilde{g}_{9,1}$	$\tilde{g}_{10,2}^{d}$				
$\tilde{g}_{5,8}$	L(3, 1)⊕2L(1, 1)	$\{k_3+at, p_3;\ m\}\oplus\{p_1;\}\oplus\{p_2;\}$	$\tilde{g}_{9,1}$	$\tilde{g}_{9,1}^{d}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{5,9}$	L(3, 4; 0)⊕2L(1, 1)	$\{j_3;\ k_1, k_2\}\oplus\{p_3;\}\oplus\{m;\}$	$\tilde{g}_{6,2}$	$\tilde{g}_{7,28}^{d}$				
$\tilde{g}_{5,10}$	L(3, 4; 0)⊕2L(1, 1)	$\{j_3;\ k_1, k_2\}\oplus\{k_3;\}\oplus\{m;\}$	$\tilde{g}_{6,2}$	$\tilde{g}_{7,28}^{d}$				
$\tilde{g}_{5,11}$	L(3, 4; 0)⊕2L(1, 1)	$\{j_3;\ k_1, k_2\}\oplus\{k_3+ap_3;\}\oplus\{m;\}$	$\tilde{g}_{6,2}$	$\tilde{g}_{6,2}^{d}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde{g}_{5,12}$	L(3, 4; 0)⊕2L(1, 1)	$\{j_3;\ p_1, p_2\}\oplus\{k_3;\}\oplus\{m;\}$	$\tilde{g}_{6,3}$	$\tilde{g}_{7,29}^{d}$				
$\tilde{g}_{5,13}$	L(3, 4; 0)⊕2L(1, 1)	$\{j_3;\ p_1, p_2\}\oplus\{k_3+at;\}\oplus\{m;\}$	$\tilde{g}_{6,4}$	$\tilde{g}_{6,4}^{d}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{5,14}$	L(3, 4; 0)⊕2L(1, 1)	$\{j_3;\ p_1, p_2\}\oplus\{t;\}\oplus\{m;\}$	$\tilde{g}_{6,1}$	$\tilde{g}_{7,19}^{d}$				
$\tilde{g}_{5,15}$	L(3, 4; 0)⊕2L(1, 1)	$\{j_3;\ p_1, p_2\}\ominus\{p_3;\}\oplus\{m;\}$	$\tilde{g}_{7,1}$	$\tilde{g}_{8,15}^{d}$				
$\tilde{g}_{5,16}$	L(3, 4; 0)⊕2L(1, 1)	$\{j_3+at;\ p_1, p_2\}\oplus\{p_3;\}\oplus\{m;\}$	$\tilde{g}_{7,1}$	$\tilde{g}_{7,1}^{d}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{5,17}$	L(3, 4; 0)⊕2L(1, 1)	$\{j_3+ap_3;\ p_1, p_2\}\oplus\{t;\}\oplus\{m;\}$	$\tilde{g}_{6,1}$	$\tilde{g}_{6,1}^{d}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde{g}_{5,18}$	L(3, 4; 0)⊕2L(1, 1)	$\{j_3+am;\ p_1, p_2\}\oplus\{p_3;\}\oplus\{t;\}$	$\tilde{g}_{6,1}$	$\tilde{g}_{9,2}^{d}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
$\tilde{g}_{5,19}$	L(3, 4; 0)⊕2L(1, 1)	$\{j_3+am;\ p_1, p_2\}\oplus\{p_3;\}$ $\oplus\{t+bm;\}$	$\tilde{g}_{6,1}$	$\tilde{g}_{6,1}^{d}$	$a\geqslant 0$ $b\neq 0$	$a\geqslant 0$ $b\neq 0$	$a\geqslant 0$ $b=\pm 1$	$a\geqslant 0$ $b=\pm 1$
$\tilde{g}_{5,20}$	L(3, 6)⊕2L(1, 1)	$\{;\ \boldsymbol{j}\}\oplus\{t;\}\oplus\{m;\}$	$\tilde{g}_{5,20}$	$\tilde{g}_{6,25}^{d}$				
$\tilde{g}_{5,21}$	L(4, 1)⊕L(1, 1)	$\{t, k_3;\ p_3, m\}\oplus\{j_3;\}$	$\tilde{g}_{5,21}$	$\tilde{g}_{6,32}^{d}$				
$\tilde{g}_{5,22}$	L(4, 1)⊕L(1, 1)	$\{t, k_2;\ p_2, m\}\oplus\{p_1;\}$	$\tilde{g}_{7,2}$	$\tilde{g}_{8,11}^{d}$				
$\tilde{g}_{5,23}$	L(4, 1)⊕L(1, 1)	$\{t, k_2+ap_3;\ p_2, m\}\oplus\{p_1;\}$	$\tilde{g}_{7,2}$	$\tilde{g}_{7,2}^{d}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{5,24}$	L(4, 1)⊕L(1, 1)	$\{k_1+bp_3, k_3+at;\ p_1, m\}\oplus\{p_2;\}$	$\tilde{g}_{7,11}$	$\tilde{g}_{7,11}^{d}$	$a>0$ $b\geqslant 0$	$a>0$ $b\geqslant 0$	$a=1$ $b\geqslant 0$	$a=1$ $b\geqslant 0$
$\tilde{g}_{5,25}$	L(4, 11)⊕L(1, 1)	$\{j_3;\ k_1+ap_2, k_2-ap_1, m\}$ $\oplus\{k_3+bp_3;\}$	$\tilde{g}_{6,5}$	$\tilde{g}_{6,5}^{d}$	$a>0$ $b\in\mathbb{R}$	$a>0$ $b\in\mathbb{R}$	$a=1$ $b\in\mathbb{R}$	$a=1$ $b\in\mathbb{R}$
$\tilde{g}_{5,26}$	L(4, 11)⊕L(1, 1)	$\{j_3;\ k_1+ap_2, k_2-ap_1, m\}\oplus\{p_3;\}$	$\tilde{g}_{6,5}$	$\tilde{g}_{6,5}^{d}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{5,27}$	L(5, 4)	$k_1+ap_2, k_2-ap_1+cp_2,$ $k_3+bp_2, p_3;\ m$ $(c, b)\neq(0, 0)$	$\tilde{g}_{7,10}$	$\tilde{g}_{7,10}^{d}$	$a>0$ $b\geqslant 0$ $c\in\mathbb{R}$	$a>0$ $b\geqslant 0$ $c\in\mathbb{R}$	$a=1$ $b\geqslant 0$ $c\in\mathbb{R}$	$a=1$ $b\geqslant 0$ $c\in\mathbb{R}$
$\tilde{g}_{5,28}$	L(5, 4)	$k_1+cp_2, k_2-cp_1+dp_2,$ $k_3+ap_1+bp_2, p_3;\ m$	$\tilde{g}_{7,10}$	$\tilde{g}_{7,10}^{d}$	$a>0$ $b>0$ $c\neq 0$ $d\in\mathbb{R}$	$a>0$ $b>0$ $c\neq 0$ $d\in\mathbb{R}$	$a=1$ $b>0$ $c\neq 0$ $d\in\mathbb{R}$	$a=1$ $b>0$ $c\neq 0$ $d\in\mathbb{R}$
$\tilde{g}_{5,29}$	L(5, 4)	$k_1+ap_2, k_2-ap_1, k_3, p_3;\ m$	$\tilde{g}_{8,7}$	$\tilde{g}_{8,7}^{d}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{5,30}$	L(5, 4)	$k_1, k_2+bp_3, p_1+ap_3, p_2;\ m$	$\tilde{g}_{7,10}$	$\tilde{g}_{7,10}^{d}$	$a>0$ $b>0$	$a>0$ $b>0$	$a>0$ $b=1$	$a>0$ $b=1$
$\tilde{g}_{5,31}$	L(5, 4)	$k_1, k_2, p_1+ap_3, p_2;\ m$	$\tilde{g}_{7,10}$	$\tilde{g}_{7,10}^{d}$	$a>0$	$a>0$	$a>0$	$a>0$
$\tilde{g}_{5,32}$	L(5, 4)	$k_1, k_2+ap_3, p_1, p_2;\ m$	$\tilde{g}_{8,3}$	$\tilde{g}_{8,3}^{d}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{5,33}$	L(5, 4)	$k_1, k_2, p_1, p_2;\ m$	$\tilde{g}_{9,1}$	$\tilde{g}_{10,2}^{d}$				
$\tilde{g}_{5,34}$	L(5, 5)	$k_3+at+ck_1, k_2+bp_3;\ p_1, p_2, m$	$\tilde{g}_{7,11}$	$\tilde{g}_{7,11}^{d}$	$a>0$ $b\in\mathbb{R}$ $c>0$	$a>0$ $b\geqslant 0$ $c>0$	$a=1$ $b\in\mathbb{R}$ $c>0$	$a=1$ $b\geqslant 0$ $c>0$

Table 3. (continued)

Number	Isomorphism class and comments	Basis	$\mathrm{nor}_{\tilde g}$	$\mathrm{nor}_{\tilde g^{d}}$	Range of parameters $\tilde G_0$	$\tilde G$	$\tilde G_0^{d}$	$\tilde G^{d}$
$\tilde g_{5,35}$	L(5, 5)	$k_1+at,\ k_2+cp_3+bt;\ p_1, p_2, m$	$\tilde g_{7,2}$	$\tilde g^{d}_{7,2}$	$a>0$ $b\geq 0$ $c\neq 0$	$a>0$ $b\geq 0$ $c>0$	$a=1$ $b\geq 0$ $c\neq 0$	$a=1$ $b\geq 0$ $c>0$
$\tilde g_{5,36}$	L(5, 13; 0)	$j_3+ak_3, p_3;\ k_1, k_2, m$	$\tilde g_{6,2}$	$\tilde g^{d}_{6,2}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{5,37}$	L(5, 13; 0)	$j_3+ak_3+bt, p_3;\ p_1, p_2, m$	$\tilde g_{7,1}$	$\tilde g^{d}_{7,1}$	$a\neq 0$ $b\geq 0$	$a>0$ $b\geq 0$	$a=\pm 1$ $b\geq 0$	$a=1$ $b\geq 0$
$\tilde g_{5,38}$	L(5,13; 0)	$j_3+ap_3,\ k_3+bp_3;\ p_1, p_2, m$	$\tilde g_{6,2}$	$\tilde g^{d}_{6,2}$	$a\neq 0$ $b\in\mathbb{R}$	$a>0$ $b\geq 0$	$a=\pm 1$ $b\in\mathbb{R}$	$a=1$ $b\geq 0$
$\tilde g_{5,39}$	L(5, 13; 0)	$j_3+ap_3,\ k_3+bt;\ p_1, p_2, m$	$\tilde g_{6,4}$	$\tilde g^{d}_{6,4}$	$a\neq 0$ $b>0$	$a\neq 0$ $b>0$	$a=\pm 1$ $b>0$	$a=\pm 1$ $b>0$
$\tilde g_{5,40}$	L(5, 13; 0)	$j_3+ap_3,\ k_3;\ p_1, p_2, m$	$\tilde g_{6,3}$	$\tilde g^{d}_{6,3}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{5,41}$	L(5, 35)	$j_3+bp_3,\ k_3+cp_3;\ k_1+ap_2,$ k_2-ap_1, m	$\tilde g_{6,5}$	$\tilde g^{d}_{6,5}$	$a>0$ $b\neq 0$ $c\in\mathbb{R}$	$a>0$ $b>0$ $c\in\mathbb{R}$	$a=1$ $b\neq 0$ $c\in\mathbb{R}$	$a=1$ $b>0$ $c\in\mathbb{R}$
$\tilde g_{5,42}$	L(5, 35)	$j_3+bk_3, p_3;\ k_1+ap_2,\ k_2-ap_1, m$	$\tilde g_{6,5}$	$\tilde g^{d}_{6,5}$	$a>0$ $b\neq 0$	$a>0$ $b>0$	$a=1$ $b\neq 0$	$a=1$ $b>0$
$\tilde g_{4,1}$	4L(1, 1)	$\{p_1;\}\oplus\{p_2;\}\oplus\{p_3;\}\oplus\{t+am;\}$	$\tilde g_{8,1}$	$\tilde g^{d}_{8,1}$	$a\neq 0$	$a\neq 0$	$a=\pm 1$	$a=\pm 1$
$\tilde g_{4,2}$	4L(1, 1)	$\{p_1;\}\oplus\{p_2;\}\oplus\{p_3;\}\oplus\{t;\}$	$\tilde g_{8,1}$	$\tilde g^{d}_{9,2}$				
$\tilde g_{4,3}$	4L(1, 1)	$\{p_1;\}\oplus\{p_2;\}\oplus\{p_3;\}\oplus\{m;\}$	$\tilde g_{11,1}$	$\tilde g^{d}_{12,1}$				
$\tilde g_{4,4}$	4L(1, 1)	$\{k_3+at;\}\oplus\{p_1;\}\oplus\{p_2;\}\oplus\{m;\}$	$\tilde g_{8,5}$	$\tilde g^{d}_{8,5}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde g_{4,5}$	4L(1, 1)	$\{k_3;\}\oplus\{p_1;\}\oplus\{p_2;\}\oplus\{m;\}$	$\tilde g_{9,1}$	$\tilde g^{d}_{10,2}$				
$\tilde g_{4,6}$	4L(1, 1)	$\{t;\}\oplus\{p_1;\}\oplus\{p_2;\}\oplus\{m;\}$	$\tilde g_{8,2}$	$\tilde g^{d}_{9,5}$				
$\tilde g_{4,7}$	4L(1, 1)	$\{k_1;\}\oplus\{k_2+ap_2;\}\oplus\{p_3;\}\oplus\{m;\}$	$\tilde g_{7,10}$	$\tilde g^{d}_{7,10}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{4,8}$	4L(1, 1)	$\{k_1;\}\oplus\{k_2;\}\oplus\{p_3;\}\oplus\{m;\}$	$\tilde g_{10,1}$	$\tilde g^{d}_{11,2}$				
$\tilde g_{4,9}$	4L(1, 1)	$\{j_3;\}\oplus\{p_3;\}\oplus\{t;\}\oplus\{m;\}$	$\tilde g_{5,21}$	$\tilde g^{d}_{6,32}$				
$\tilde g_4'$	4L(1, 1)	cf table 4						
$\tilde g_{4,10}$	L(3, 1)⊕L(1, 1)	$\{k_1, p_1;\ m\}\oplus\{k_2;\}$	$\tilde g_{7,10}$	$\tilde g^{d}_{8,13}$				
$\tilde g_{4,11}$	L(3, 1)⊕L(1, 1)	$\{k_1+bp_2+ap_3, p_1;\ m\}$ $\oplus\{k_2+bp_1;\}$	$\tilde g_{7,10}$	$\tilde g^{d}_{7,10}$	$a>0$ $b\geq 0$	$a>0$ $b\geq 0$	$a=1$ $b\geq 0$	$a=1$ $b\geq 0$
$\tilde g_{4,12}$	L(3, 1)⊕L(1, 1)	$\{k_1+ap_2,\ k_2-ap_1;\ m\}$ $\oplus\{k_2+ap_1;\}$	$\tilde g_{7,10}$	$\tilde g^{d}_{7,10}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde g_{4,13}$	L(3, 1)⊕L(1, 1)	$\{k_1+bp_2+cp_3, p_1;\ m\}$ $\oplus\{k_2+ap_3+bp_1;\}$	$\tilde g_{7,10}$	$\tilde g^{d}_{7,10}$	$a>0$ $b\geq 0$ $c\in\mathbb{R}$	$a>0$ $b\geq 0$ $c\in\mathbb{R}$	$a=1$ $b\geq 0$ $c\in\mathbb{R}$	$a=1$ $b\geq 0$ $c\in\mathbb{R}$
$\tilde g_{4,14}$	L(3, 1)⊕L(1, 1)	$\{k_1,\ p_3+ap_1;\ m\}\oplus\{k_2;\}$	$\tilde g_{7,10}$	$\tilde g^{d}_{8,13}$	$a>0$	$a>0$	$a>0$	$a>0$
$\tilde g_{4,15}$	L(3, 1)⊕L(1, 1)	$\{k_1,\ p_3+ap_1;\ m\}\oplus\{k_2+bp_2;\}$	$\tilde g_{7,10}$	$\tilde g^{d}_{7,10}$	$a>0$ $b\neq 0$	$a>0$ $b>0$	$a>0$ $b=\pm 1$	$a>0$ $b=1$
$\tilde g_{4,16}$	L(3, 1)⊕L(1, 1)	$k_1,\ k_2+bp_1+cp_2,\ p_3+ap_1, m$	$\tilde g_{7,10}$	$\tilde g^{d}_{7,10}$	$a>0$ $b>0$ $c\in\mathbb{R}$	$a>0$ $b>0$ $c\in\mathbb{R}$	$a>0$ $b=1$ $c\in\mathbb{R}$	$a>0$ $b=1$ $c\in\mathbb{R}$
$\tilde g_{4,17}$	L(3, 1)⊕L(1, 1)	$k_1+bp_2,\ k_2+cp_1+dp_2,$ p_3+ap_1, m	$\tilde g_{7,10}$	$\tilde g^{d}_{7,10}$	$a>0$ $b>0$ $c, d\in\mathbb{R}$	$a>0$ $b>0$ $c, d\in\mathbb{R}$	$a>0$ $b=1$ $c, d\in\mathbb{R}$	$a>0$ $b=1$ $c, d\in\mathbb{R}$
$\tilde g_{4,18}$	4L(1, 1)	$\{k_1+ap_2,\ k_2-ap_1;\ m\}\oplus\{p_3;\}$	$\tilde g_{8,7}$	$\tilde g^{d}_{8,7}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde g_{4,19}$	4L(1, 1)	$\{k_1+ap_2,\ k_2-ap_1+bp_2;\ m\}$ $\oplus\{p_3;\}$	$\tilde g_{7,10}$	$\tilde g^{d}_{7,10}$	$a>0$ $b\neq 0$	$a>0$ $b\neq 0$	$a=1$ $b\neq 0$	$a=1$ $b\neq 0$
$\tilde g_{4,20}$	4L(1, 1)	$\{k_3+at+bk_2,\ p_2;\ m\}\oplus\{p_1;\}$	$\tilde g_{7,11}$	$\tilde g^{d}_{7,11}$	$a>0$ $b>0$	$a>0$ $b>0$	$a=1$ $b>0$	$a=1$ $b>0$
$\tilde g_{4,21}$	4L(1, 1)	$\{k_2+at,\ p_2;\ m\}\oplus\{p_1;\}$	$\tilde g_{7,2}$	$\tilde g^{d}_{7,2}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde g_{4,22}$	4L(1, 1)	$\{k_3,\ p_3+ap_2;\ m\}\oplus\{p_1;\}$	$\tilde g_{7,10}$	$\tilde g^{d}_{8,13}$	$a>0$	$a>0$	$a>0$	$a>0$
$\tilde g_{4,23}$	4L(1, 1)	$\{k_3, p_3;\ m\}\oplus\{p_1;\}$	$\tilde g_{8,3}$	$\tilde g^{d}_{9,3}$				
$\tilde g_{4,24}$	4L(1, 1)	$\{k_3+ap_1,\ p_3;\ m\}\oplus\{p_2;\}$	$\tilde g_{8,3}$	$\tilde g^{d}_{8,3}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde g_{4,25}$	4L(1, 1)	$\{k_3+at,\ p_3;\ m\}\oplus\{j_3;\}$	$\tilde g_{5,21}$	$\tilde g^{d}_{5,21}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde g_{4,26}$	4L(1, 1)	$\{k_3, p_3;\ m\}\oplus\{j_3;\}$	$\tilde g_{5,21}$	$\tilde g^{d}_{6,32}$				
$\tilde g_4''$	4L(1, 1)	cf table 4						
$\tilde g_{4,27}$	L(3, 4; 0)⊕L(1, 1)	$\{j_3;\ k_1, k_2\}\oplus\{m;\}$	$\tilde g_{6,2}$	$\tilde g^{d}_{7,28}$				
$\tilde g_{4,28}$	L(3, 4; 0)⊕L(1, 1)	$\{j_3;\ +ap_3;\ k_1, k_2\}\oplus\{m;\}$	$\tilde g_{6,2}$	$\tilde g^{d}_{6,2}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$

Table 3. (continued)

Number	Isomorphism class and comments	Basis	$\mathrm{nor}_{\tilde{g}}$	$\mathrm{nor}_{\tilde{g}^d}$	Range of parameters			
					$\tilde{G}_0$	$\tilde{G}$	$\tilde{G}_0^d$	$\tilde{G}^d$
$\tilde{g}_{4,29}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+ak_3+bp_3;\ k_1, k_2\}\oplus\{m;\}$	$\tilde{g}_{6,2}$	$\tilde{g}^d_{6,2}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
					$b\in\mathbb{R}$	$b\in\mathbb{R}$	$b\in\mathbb{R}$	$b\in\mathbb{R}$
$\tilde{g}_{4,30}$	$L(3,4;0)\oplus L(1,1)$	$\{k_3+at+bj_3;\ p_1, p_2\}\oplus\{m;\}$	$\tilde{g}_{6,4}$	$\tilde{g}^d_{6,4}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\neq 0$	$b>0$	$b\neq 0$	$b>0$
$\tilde{g}_{4,31}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+at;\ p_1, p_2\}\oplus\{m;\}$	$\tilde{g}_{8,2}$	$\tilde{g}^d_{8,2}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{4,32}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+ap_3;\ p_1, p_2\}\oplus\{m;\}$	$\tilde{g}_{7,1}$	$\tilde{g}^d_{7,1}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde{g}_{4,33}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3;\ p_1, p_2\}\oplus\{m;\}$	$\tilde{g}_{9,1}$	$\tilde{g}^d_{9,1}$				
$\tilde{g}_{4,34}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+am;\ p_1, p_2\}\oplus\{t;\}$	$\tilde{g}_{6,1}$	$\tilde{g}^d_{7,19}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
$\tilde{g}_{4,35}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+bp_3+am;\ p_1, p_2\}\oplus\{t+cm;\}$	$\tilde{g}_{6,1}$	$\tilde{g}^d_{6,1}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
					$b\neq 0$	$b>0$	$b=\pm 1$	$b=1$
					$c\in\mathbb{R}$	$c\in\mathbb{R}$	$c\in\mathbb{R}$	$c\in\mathbb{R}$
$\tilde{g}_{4,36}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+am;\ p_1, p_2\}\oplus\{t+cm;\}$	$\tilde{g}_{6,1}$	$\tilde{g}^d_{6,1}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
					$c\neq 0$	$c\neq 0$	$c\neq 0$	$c\neq 0$
$\tilde{g}_{4,37}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+ak_3;\ p_1, p_2\}\oplus\{m;\}$	$\tilde{g}_{6,3}$	$\tilde{g}^d_{6,3}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde{g}_{4,38}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+am;\ k_1, k_2\}\oplus\{k_3;\}$	$\tilde{g}_{5,10}$	$\tilde{g}^d_{6,26}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
$\tilde{g}_{4,39}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+bm;\ k_1, k_2\}\oplus\{k_3+ap_3;\}$	$\tilde{g}_{5,10}$	$\tilde{g}^d_{5,10}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
					$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$
$\tilde{g}_{4,40}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+am;\ k_1, k_2\}\oplus\{p_3;\}$	$\tilde{g}_{5,9}$	$\tilde{g}^d_{6,28}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
$\tilde{g}_{4,41}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+bm;\ p_1, p_2\}\oplus\{k_3+at;\}$	$\tilde{g}_{5,13}$	$\tilde{g}^d_{5,13}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\in\mathbb{R}$	$b\in\mathbb{R}$	$b\in\mathbb{R}$	$b\in\mathbb{R}$
$\tilde{g}_{4,42}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+am;\ p_1, p_2\}\oplus\{k_3;\}$	$\tilde{g}_{5,12}$	$\tilde{g}^d_{6,29}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
$\tilde{g}_{4,43}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+am;\ p_1, p_2\}\oplus\{p_3;\}$	$\tilde{g}_{6,1}$	$\tilde{g}^d_{7,19}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
$\tilde{g}_{4,44}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+at;\ p_1, p_2\}\oplus\{p_3;\}$	$\tilde{g}_{6,1}$	$\tilde{g}^d_{6,1}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{4,45}$	$L(3,4;0)\oplus L(1,1)$	$\{j_3+bt+am;\ p_1, p_2\}\oplus\{p_3;\}$	$\tilde{g}_{6,1}$	$\tilde{g}^d_{6,1}$	$a>0$	$a>0$	$a>0$	$a>0$
					$b\neq 0$	$b\neq 0$	$b=\pm 1$	$b=\pm 1$
$\tilde{g}_{4,46}$	$L(3,6)\oplus L(1,1)$	$\{;j\}\oplus\{m;\}$	$\tilde{g}_{5,20}$	$\tilde{g}^d_{6,25}$				
$\tilde{g}_{4,47}$	$L(3,6)\oplus L(1,1)$	$\{;j\}\oplus\{t;\}$	$\tilde{g}_{5,20}$	$\tilde{g}^d_{6,25}$				
$\tilde{g}_{4,48}$	$L(3,6)\oplus L(1,1)$	$\{;j\}\oplus\{t+am\}$	$\tilde{g}_{5,20}$	$\tilde{g}^d_{5,20}$	$a\neq 0$	$a\neq 0$	$a=\pm 1$	$a=\pm 1$
$\tilde{g}_{4,49}$	$L(4,1)$	$k_3+at,\ k_1+bp_3+cp_2;\ p_1, m$	$\tilde{g}_{6,7}$	$\tilde{g}^d_{6,7}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$
					$c\in\mathbb{R}$	$c\in\mathbb{R}$	$c\in\mathbb{R}$	$c\in\mathbb{R}$
$\tilde{g}_{4,50}$	$L(4,1)$	$t,\ k_3+ap_1;\ p_3, m$	$\tilde{g}_{6,6}$	$\tilde{g}^d_{6,6}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{4,51}$	$L(4,1)$	$t,\ k_3;\ p_3, m$	$\tilde{g}_{7,1}$	$\tilde{g}^d_{8,15}$				
$\tilde{g}_{4,52}$	$L(4,1)$	$j_3+at,\ k_3+bt;\ p_3, m$	$\tilde{g}_{5,21}$	$\tilde{g}^d_{5,21}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\in\mathbb{R}$	$b\geqslant 0$	$b\in\mathbb{R}$	$b\geqslant 0$
$\tilde{g}_{4,53}$	$L(4,1)$	$t,\ j_3+ak_3;\ p_3, m$	$\tilde{g}_{5,21}$	$\tilde{g}^d_{5,21}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde{g}_{4,54}$	$L(4,11)$	$j_3+bk_3+cp_3;\ k_1+ap_2,$ k_2-ap_1, m	$\tilde{g}_{6,5}$	$\tilde{g}^d_{6,5}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\neq 0$	$b>0$	$b\neq 0$	$b>0$
					$c\in\mathbb{R}$	$c\in\mathbb{R}$	$c\in\mathbb{R}$	$c\in\mathbb{R}$
$\tilde{g}_{4,55}$	$L(4,11)$	$j_3+cp_3;\ k_1+ap_2,\ k_2-ap_1, m$	$\tilde{g}_{6,5}$	$\tilde{g}^d_{6,5}$	$a>0$	$a>0$	$a=1$	$a=1$
					$c\in\mathbb{R}$	$c\geqslant 0$	$c\in\mathbb{R}$	$c\geqslant 0$
$\tilde{g}_{3,1}$	$3L(1,1)$	$\{k_1;\}\oplus\{k_2+ap_2;\}\oplus\{m;\}$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde{g}_{3,2}$	$3L(1,1)$	$\{k_1+bp_2;\}\oplus\{k_2+bp_1+cp_2$ $+ap_3;\}\oplus\{m;\}$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$
					$c\in\mathbb{R}$	$c\in\mathbb{R}$	$c\in\mathbb{R}$	$c\in\mathbb{R}$
$\tilde{g}_{3,3}$	$3L(1,1)$	$\{k_1;\}\oplus\{k_2;\}\oplus\{m;\}$	$\tilde{g}_{8,7}$	$\tilde{g}^d_{9,6}$				
$\tilde{g}_{3,4}$	$3L(1,1)$	$\{j_3+am;\}\oplus\{p_3;\}\oplus\{t;\}$	$\tilde{g}_{4,9}$	$\tilde{g}^d_{5,42}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
$\tilde{g}_{3,5}$	$3L(1,1)$	$\{j_3+am;\}\oplus\{p_3;\}\oplus\{t+bm;\}$	$\tilde{g}_{4,9}$	$\tilde{g}^d_{4,9}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
					$b\neq 0$	$b\neq 0$	$b=\pm 1$	$b=\pm 1$
$\tilde{g}_{3,6}$	$3L(1,1)$	$\{k_3+at;\}\oplus\{p_1;\}\oplus\{p_2;\}$	$\tilde{g}_{5,13}$	$\tilde{g}^d_{5,13}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{3,7}$	$3L(1,1)$	$\{k_3;\}\oplus\{p_1;\}\oplus\{p_2;\}$	$\tilde{g}_{5,14}$	$\tilde{g}^d_{6,31}$				
$\tilde{g}_{3,8}$	$3L(1,1)$	$\{t;\}\oplus\{p_1;\}\oplus\{p_2;\}$	$\tilde{g}_{6,1}$	$\tilde{g}^d_{7,19}$				
$\tilde{g}_{3,9}$	$3L(1,1)$	$\{t+am;\}\oplus\{p_1;\}\oplus\{p_2;\}$	$\tilde{g}_{6,1}$	$\tilde{g}^d_{6,1}$	$a\neq 0$	$a\neq 0$	$a=\pm 1$	$a=\pm 1$
$\tilde{g}_{3,10}$	$3L(1,1)$	$\{j_3;\}\oplus\{k_3+at;\}\oplus\{m;\}$	$\tilde{g}_{4,25}$	$\tilde{g}^d_{4,25}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{3,11}$	$3L(1,1)$	$\{j_3;\}\oplus\{t;\}\oplus\{m;\}$	$\tilde{g}_{4,9}$	$\tilde{g}^d_{5,42}$				
$\tilde{g}_{3,12}$	$3L(1,1)$	$\{j_3;\}\oplus\{k_3;\}\oplus\{m;\}$	$\tilde{g}_{4,26}$	$\tilde{g}^d_{5,55}$				
$\tilde{g}_{3,13}$	$3L(1,1)$	$\{j_3;\}\oplus\{p_3;\}\oplus\{m;\}$	$\tilde{g}_{5,21}$	$\tilde{g}^d_{6,32}$				
$\tilde{g}_{3,14}$	$3L(1,1)$	$\{j_3+at;\}\oplus\{p_3;\}\oplus\{m;\}$	$\tilde{g}_{5,21}$	$\tilde{g}^d_{5,21}$	$a>0$	$a>0$	$a=1$	$a=1$

Table 3. (continued)

Number	Isomorphism class and comments	Basis	$\text{nor}_{\tilde g}$	$\text{nor}_{\tilde g^d}$	Range of parameters $\tilde G_0$	$\tilde G$	$\tilde G_0^d$	$\tilde G^d$
$\tilde g_{3,15}$	3L(1, 1)	$\{k_3;\}\oplus\{p_1;\}\oplus\{m;\}$	$\tilde g_{7,10}$	$\tilde g^d_{8,13}$				
$\tilde g_{3,16}$	3L(1, 1)	$\{k_3+ap_2;\}\oplus\{p_1;\}\oplus\{m;\}$	$\tilde g_{7,10}$	$\tilde g^d_{7,10}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde g_{3,17}$	3L(1, 1)	$\{p_1;\}\oplus\{p_2;\}\oplus\{m;\}$	$\tilde g_{9,1}$	$\tilde g^d_{10,2}$				
$\tilde g_{3,18}$	3L(1, 1)	$\{k_3+at;\}\oplus\{p_2;\}\oplus\{m;\}$	$\tilde g_{6,7}$	$\tilde g^d_{6,7}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde g_{3,19}$	3L(1, 1)	$\{t;\}\oplus\{p_3;\}\oplus\{m;\}$	$\tilde g_{6,6}$	$\tilde g^d_{7,20}$				
$\tilde g_{3,20}$	3L(1, 1)	$\{j_3+ap_3;\}\oplus\{t;\}\oplus\{m;\}$	$\tilde g_{4,9}$	$\tilde g^d_{4,9}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{3,21}$	3L(1, 1)	$\{k_1;\}\oplus\{k_2;\}\oplus\{k_3;\}$	$\tilde g_{5,10}$	$\tilde g^d_{6,26}$				
$\tilde g_{3,22}$	3L(1, 1)	$\{k_1;\}\oplus\{k_2;\}\oplus\{k_3+bp_3;\}$	$\tilde g_{5,11}$	$\tilde g^d_{5,11}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{3,23}$	3L(1, 1)	$\{k_1+ap_1;\}\oplus\{k_2+ap_2;\}\oplus\{k_3;\}$	$\tilde g'_{4,1}$	$\tilde g'^d_{4,1}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
		$a\neq b\neq 0$						
$\tilde g_{3,24}$	3L(1, 1)	$\{k_1+ap_1;\}\oplus\{k_2;\}\oplus\{k_3;\}$	$g'_{4,3}$	$\tilde g'^d_{4,3}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{3,25}$	3L(1, 1)	$\{k_1;\}\oplus\{k_2+ap_2;\}\oplus\{p_3;\}$	$\tilde g_{4,7}$	$\tilde g^d_{4,7}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{3,26}$	3L(1, 1)	$\{k_1;\}\oplus\{k_2;\}\oplus\{p_3;\}$	$\tilde g_{5,9}$	$\tilde g^d_{6,28}$				
$\tilde g_{3,27}$	3L(1, 1)	$\{p_1;\}\oplus\{p_2;\}\oplus\{p_3;\}$	$\tilde g_{8,1}$	$\tilde g^d_{9,2}$				
$\tilde g_{3,28}$	L(3, 1)	$k_1,\ k_2+ap_1+cp_2+bp_3;\ m$	$\tilde g_{7,10}$	$\tilde g^d_{7,10}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
					$b>0$	$b>0$	$b=1$	$b=1$
					$c\in\mathbb{R}$	$c\in\mathbb{R}$	$c\in\mathbb{R}$	$c\in\mathbb{R}$
$\tilde g_{3,29}$	L(3, 1)	$k_1+ap_2,\ k_2+bp_1+cp_2+dp_3;\ m$	$\tilde g_{7,10}$	$\tilde g^d_{7,10}$	$a>0$	$a>0$	$a>0$	$a>0$
		$a\neq b$			$b, c\in\mathbb{R}$	$b, c\in\mathbb{R}$	$b, c\in\mathbb{R}$	$b, c\in\mathbb{R}$
					$d>0$	$d>0$	$d=1$	$d=1$
$\tilde g_{3,30}$	L(3, 1)	$k_1+ap_2,\ k_2-ap_1+bp_2;\ m$	$\tilde g_{7,10}$	$\tilde g^d_{7,10}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\neq 0$	$b\neq 0$	$b\neq 0$	$b\neq 0$
$\tilde g_{3,31}$	L(3, 1)	$k_1+ap_2,\ k_2-ap_1;\ m$	$\tilde g_{8,7}$	$\tilde g^d_{8,7}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde g_{3,32}$	L(3, 1)	$j_3+ak_3+bt,\ p_3;\ m$	$\tilde g_{5,21}$	$\tilde g^d_{5,21}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
					$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$
$\tilde g_{3,33}$	L(3, 1)	$k_3,\ p_3;\ m$	$\tilde g_{9,1}$	$\tilde g^d_{9,6}$				
$\tilde g_{3,34}$	L(3, 1)	$k_3+at,\ p_3;\ m$	$\tilde g_{7,1}$	$\tilde g^d_{7,1}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde g_{3,35}$	L(3, 1)	$k_2,\ p_2+ap_1,\ m$	$\tilde g_{6,8}$	$\tilde g^d_{6,8}$	$a>0$	$a>0$	$a>0$	$a>0$
$\tilde g_{3,36}$	L(3, 1)	$k_3+at+bk_2,\ p_2;\ m$	$\tilde g_{6,7}$	$\tilde g^d_{6,7}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b>0$	$b>0$	$b>0$	$b>0$
$\tilde g_{3,37}$	L(3, 1)	$k_3+ap_1,\ p_3+bp_2;\ m$	$\tilde g_{7,10}$	$\tilde g^d_{7,10}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b>0$	$b>0$	$b>0$	$b>0$
$\tilde g_{3,38}$	L(3, 1)	$k_3+ap_1,\ p_3,\ m$	$\tilde g_{8,3}$	$\tilde g^d_{8,3}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde g_{3,39}$	L(3, 1)	$j_3+bp_3,\ k_3+at;\ m$	$\tilde g_{4,25}$	$\tilde g^d_{4,25}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\neq 0$	$b>0$	$b\neq 0$	$b>0$
$\tilde g_{3,40}$	L(3, 1)	$j_3+ap_3,\ k_3;\ m$	$\tilde g_{5,21}$	$\tilde g^d_{5,21}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{3,41}$	L(3, 4; 0)	$j_3+ak_3,\ k_1,\ k_2$	$\tilde g_{5,10}$	$\tilde g^d_{5,10}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{3,42}$	L(3, 4; 0)	$k_3+ap_3+bj_3;\ k_1,\ k_2$	$\tilde g_{5,11}$	$\tilde g^d_{5,11}$	$a\neq 0$	$a\neq 0$	$a=\neq 1$	$a=\pm 1$
					$b\neq 0$	$b>0$	$b\neq 0$	$b>0$
$\tilde g_{3,43}$	L(3, 4; 0)	$j_3;\ k_1,\ k_2$	$\tilde g_{5,9}$	$\tilde g^d_{6,28}$				
$\tilde g_{3,44}$	L(3, 4; 0)	$j_3+ap_3;\ k_1,\ k_2$	$\tilde g_{5,9}$	$\tilde g^d_{5,9}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{3,45}$	L(3, 4; 0)	$j_3+ak_3;\ k_1,\ k_2$	$\tilde g_{5,12}$	$\tilde g^d_{5,12}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{3,46}$	L(3, 4; 0)	$j_3;\ p_1,\ p_2$	$\tilde g_{7,1}$	$\tilde g^d_{8,15}$				
$\tilde g_{3,47}$	L(3, 4; 0)	$j_3+at+bm;\ p_1,\ p_2$	$\tilde g_{6,1}$	$\tilde g^d_{6,1}$	$a\neq 0$	$a\neq 0$	$a=\pm 1$	$a=\pm 1$
					$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$
$\tilde g_{3,48}$	L(3, 4; 0)	$j_3+am;\ p_1,\ p_2$	$\tilde g_{7,1}$	$\tilde g^d_{8,15}$	$a>0$	$a>0$	$a>0$	$a>0$
$\tilde g_{3,49}$	L(3, 4; 0)	$j_3+ap_3;\ p_1,\ p_2$	$\tilde g_{6,1}$	$\tilde g^d_{6,1}$	$a\neq 0$	$a>0$	$a=\pm 1$	$a=1$
$\tilde g_{3,50}$	L(3, 4; 0)	$k_3+at+bj_3;\ p_1,\ p_2$	$\tilde g_{5,13}$	$\tilde g^d_{5,13}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\neq 0$	$b>0$	$b\neq 0$	$b>0$
$\tilde g_{3,51}$	L(3, 4; 0)	$j_3+am;\ k_1,\ k_2$	$\tilde g_{6,2}$	$\tilde g^d_{7,28}$	$a>0$	$a>0$	$a>0$	$a>0$
$\tilde g_{3,52}$	L(3, 6)	$;j$	$\tilde g_{5,20}$	$\tilde g^d_{6,25}$				
$\tilde g_{2,1}$	2L(1, 1)	$\{j_3+am;\}\oplus\{t;\}$	$\tilde g_{4,9}$	$\tilde g^d_{5,42}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
$\tilde g_{2,2}$	2L(1, 1)	$\{j_3+cp_3+bm;\}\oplus\{t+am;\}$	$\tilde g_{4,9}$	$\tilde g^d_{4,9}$	$a\neq 0$	$a\neq 0$	$a=\pm 1$	$a=\pm 1$
					$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$	$b\geqslant 0$
					$c\in\mathbb{R}$	$c\geqslant 0$	$c\in\mathbb{R}$	$c\geqslant 0$
$\tilde g_{2,3}$	2L(1, 1)	$\{j_3+bp_3+am;\}\oplus\{t;\}$	$\tilde g_{4,9}$	$\tilde g^d_{4,9}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$
					$b\neq 0$	$b>0$	$b=\pm 1$	$b=1$
$\tilde g_{2,4}$	2L(1, 1)	$\{j_3+am;\}\oplus\{p_3;\}$	$\tilde g_{4,9}$	$\tilde g^d_{5,42}$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$	$a\geqslant 0$

Table 3. (continued)

Number	Isomorphism class and comments	Basis	$\mathrm{nor}_{\tilde{g}}$	$\mathrm{nor}_{\tilde{g}^d}$	Range of parameters $\tilde{G}_0$	$\tilde{G}$	$\tilde{G}_0^d$	$\tilde{G}^d$
$\tilde{g}_{2,5}$	2L(1, 1)	$\{j_3+at;\}\oplus\{p_3;\}$	$\tilde{g}_{4,9}$	$\tilde{g}^d_{4,9}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{2,6}$	2L(1, 1)	$\{j_3+bt+am;\}\oplus\{p_3;\}$	$\tilde{g}_{4,9}$	$\tilde{g}^d_{4,9}$	$a>0$	$a>0$	$a>0$	$a>0$
					$b\in\mathbb{R}$	$b\in\mathbb{R}$	$b=0,\pm1$	$b=0,\pm1$
$\tilde{g}_{2,7}$	2L(1, 1)	$\{k_2+ap_1;\}\oplus\{p_3;\}$	$\tilde{g}_{5,2}$	$\tilde{g}^d_{5,2}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{2,8}$	2L(1, 1)	$\{k_2;\}\oplus\{p_3;\}$	$\tilde{g}_{5,6}$	$\tilde{g}^d_{6,36}$				
$\tilde{g}_{2,9}$	2L(1, 1)	$\{k_3+at;\}\oplus\{p_1;\}$	$\tilde{g}_{4,4}$	$\tilde{g}^d_{4,4}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{2,10}$	2L(1, 1)	$\{k_1;\}\oplus\{k_2;\}$	$\tilde{g}_{6,2}$	$\tilde{g}^d_{7,28}$				
$\tilde{g}_{2,11}$	2L(1, 1)	$\{k_1+bp_2;\}\oplus$	$\tilde{g}_{5,3}$	$\tilde{g}^d_{5,3}$	$a>0$	$a>0$	$a=1$	$a=1$
		$\{k_2+bp_1+cp_2+ap_3;\}$			$b\geqslant0$	$b\geqslant0$	$b\geqslant0$	$b\geqslant0$
					$c\in\mathbb{R}$	$c\geqslant0$	$c\in\mathbb{R}$	$c\geqslant0$
$\tilde{g}_{2,12}$	2L(1, 1)	$\{k_1;\}\oplus\{k_2+ap_2;\}$	$\tilde{g}_{5,4}$	$\tilde{g}^d_{5,4}$	$a\neq0$	$a>0$	$a=\pm1$	$a=1$
$\tilde{g}_{2,13}$	2L(1, 1)	$\{j_3+bm;\}\oplus\{k_3+at;\}$	$\tilde{g}_{4,25}$	$\tilde{g}^d_{4,25}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\in\mathbb{R}$	$b\in\mathbb{R}$	$b\in\mathbb{R}$	$b\in\mathbb{R}$
$\tilde{g}_{2,14}$	2L(1, 1)	$\{j_3+am;\}\oplus\{k_3;\}$	$\tilde{g}_{4,26}$	$\tilde{g}^d_{5,55}$	$a\geqslant0$	$a\geqslant0$	$a\geqslant0$	$a\geqslant0$
$\tilde{g}_{2,15}$	2L(1, 1)	$\{t+am;\}\oplus\{p_3;\}$	$\tilde{g}_{6,1}$	$\tilde{g}^d_{6,1}$	$a\neq0$	$a\neq0$	$a=\pm1$	$a=\pm1$
$\tilde{g}_{2,16}$	2L(1, 1)	$\{t;\}\oplus\{p_3;\}$	$\tilde{g}_{6,1}$	$\tilde{g}^d_{7,19}$				
$\tilde{g}_{2,17}$	2L(1, 1)	$\{p_3;\}\oplus\{m;\}$	$\tilde{g}_{9,1}$	$\tilde{g}^d_{10,2}$				
$\tilde{g}_{2,18}$	2L(1, 1)	$\{k_3+at+bj_3;\}\oplus\{m;\}$	$\tilde{g}_{4,25}$	$\tilde{g}^d_{4,25}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\neq0$	$b>0$	$b\neq0$	$b>0$
$\tilde{g}_{2,19}$	2L(1, 1)	$\{j_3+at;\}\oplus\{m;\}$	$\tilde{g}_{5,21}$	$\tilde{g}^d_{5,21}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{2,20}$	2L(1, 1)	$\{j_3;\}\oplus\{m;\}$	$\tilde{g}_{5,21}$	$\tilde{g}^d_{6,32}$				
$\tilde{g}_{2,21}$	2L(1, 1)	$\{j_3+ak_3;\}\oplus\{m;\}$	$\tilde{g}_{4,26}$	$\tilde{g}^d_{4,26}$	$a\neq0$	$a>0$	$a=\pm1$	$a=1$
$\tilde{g}_{2,22}$	2L(1, 1)	$\{j_3+ap_3;\}\oplus\{m;\}$	$\tilde{g}_{5,21}$	$\tilde{g}_{5,21}$	$a\neq0$	$a>0$	$a=\pm1$	$a=1$
$\tilde{g}_{2,23}$	2L(1, 1)	$\{k_3+at;\}\oplus\{m;\}$	$\tilde{g}_{6,4}$	$\tilde{g}^d_{6,4}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{2,24}$	2L(1, 1)	$\{k_3;\}\oplus\{m;\}$	$\tilde{g}_{8,7}$	$\tilde{g}^d_{9,6}$				
$\tilde{g}_{2,25}$	2L(1, 1)	$\{t;\}\oplus\{m;\}$	$\tilde{g}_{8,1}$	$\tilde{g}^d_{9,2}$				
$\tilde{g}_{2,26}$	2L(1, 1)	$\{k_3+ap_1;\}\oplus\{m;\}$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$	$a\neq0$	$a>0$	$a=\pm1$	$a=1$
$\tilde{g}_{2,27}$	2L(1, 1)	$\{p_1;\}\oplus\{p_2;\}$	$\tilde{g}_{7,1}$	$\tilde{g}^d_{8,15}$				
$\tilde{g}_{1,1}$	L(1, 1)	$j_3+am;$	$\tilde{g}_{5,21}$	$\tilde{g}^d_{6,32}$	$a\geqslant0$	$a\geqslant0$	$a\geqslant0$	$a\geqslant0$
$\tilde{g}_{1,2}$	L(1, 1)	$k_3+at+bj_3;$	$\tilde{g}_{3,10}$	$\tilde{g}^d_{3,10}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\neq0$	$b>0$	$b\neq0$	$b>0$
$\tilde{g}_{1,3}$	L(1, 1)	$j_3+at+bm;$	$\tilde{g}_{4,9}$	$\tilde{g}^d_{4,9}$	$a>0$	$a>0$	$a=1$	$a=1$
					$b\in\mathbb{R}$	$b\in\mathbb{R}$	$b\in\mathbb{R}$	$b\in\mathbb{R}$
$\tilde{g}_{1,4}$	L(1, 1)	$j_3+ak_3;$	$\tilde{g}_{4,26}$	$\tilde{g}^d_{4,26}$	$a\neq0$	$a>0$	$a=\pm1$	$a=1$
$\tilde{g}_{1,5}$	L(1, 1)	$p_3+aj_3;$	$\tilde{g}_{4,9}$	$\tilde{g}^d_{4,9}$	$a\neq0$	$a>0$	$a=\pm1$	$a=1$
$\tilde{g}_{1,6}$	L(1, 1)	$p_3;$	$\tilde{g}_{8,2}$	$\tilde{g}^d_{9,5}$				
$\tilde{g}_{1,7}$	L(1, 1)	$t;$	$\tilde{g}_{8,1}$	$\tilde{g}^d_{9,2}$				
$\tilde{g}_{1,8}$	L(1, 1)	$t+am;$	$\tilde{g}_{8,1}$	$\tilde{g}^d_{8,1}$	$a\neq0$	$a\neq0$	$a=\pm1$	$a=\pm1$
$\tilde{g}_{1,9}$	L(1, 1)	$m;$	$\tilde{g}_{11,1}$	$\tilde{g}^d_{12,1}$				
$\tilde{g}_{1,10}$	L(1, 1)	$k_3+at;$	$\tilde{g}_{5,13}$	$\tilde{g}^d_{5,13}$	$a>0$	$a>0$	$a=1$	$a=1$
$\tilde{g}_{1,11}$	L(1, 1)	$k_3;$	$\tilde{g}_{7,7}$	$\tilde{g}^d_{8,16}$				
$\tilde{g}_{1,12}$	L(1, 1)	$k_3+ap_1;$	$\tilde{g}_{6,10}$	$\tilde{g}^d_{6,10}$	$a\neq0$	$a>0$	$a=\pm1$	$a=1$

The similitude algebra is viewed as the semidirect sum

$$\tilde{g}^d=\{d\}\oplus\tilde{g} \tag{3.6}$$

i.e. the factor algebra $f\sim\{d\}$ is one dimensional and the ideal $n\sim\tilde{g}$ is eleven dimensional and, of course, non-Abelian (and not solvable). The subalgebras of f are hence $f_1\sim\{d\}$ and $f_2\sim\{\varnothing\}$. The classification is performed under the connected component of the extended Galilei-similitude group $\tilde{G}_0^d$ and under the group $\tilde{G}^d$, including time reversal and parity.

The following three types of subalgebras of $\tilde{g}^d$ exist.

(1) Subalgebras obtained from the subalgebra $f_2\sim\{\varnothing\}$ of the factor algebra. These are subalgebras of the extended Galilei algebra $\tilde{g}$. A list of representatives of $\tilde{G}_0^d$ and

Table 4. Classification of subalgebras of the form $k_1+a_1p_1+a_2p_2+a_3p_3$, $k_2-a_2p_1+b_1p_2+b_2p_3$, $k_3-a_3p_1-b_2p_2$, m under $\tilde{G}_0$, $\tilde{G}$, $\tilde{G}_0^d$ and $\tilde{G}^d$. No indication on the range of a parameter means $\neq 0$. Indication in brackets ( ) stands for classification under $\tilde{G}$ and $\tilde{G}^d$.

Number	Isomorphism class	Range of parameters: Under $\tilde{G}_0$ and $\tilde{G}$: a_1	a_2	a_3	b_1	b_2	Under $\tilde{G}_0^d$ and $\tilde{G}^d$	$\mathrm{nor}_{\tilde{g}}$	$\mathrm{nor}_{\tilde{g}}^d$
$\tilde{g}''_{4,1}$	L(3, 1)⊕L(1, 1)		>			>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,2}$	L(3, 1)⊕L(1, 1)	0	>			>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,3}$	L(3, 1)⊕L(1, 1)		0	>		>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,4}$	L(3, 1)⊕L(1, 1)	0	0	>		>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,5}$	L(3, 1)⊕L(1, 1)		>	0		>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,6}$	L(3, 1)⊕L(1, i)	0	>	0		>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,7}$	L(3, 1)⊕L(1, 1)		0	0		>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,8}$	L(3, 1)⊕L(1, 1)	0	0	0		>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,9}$	L(3, 1)⊕L(1, 1)		>		0	>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,10}$	L(3, 1)⊕L(1, 1)	0	>		0	>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,11}$	L(3, 1)⊕L(1, 1)		0	>	0	>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,12}$	L(3, 1)⊕L(1, 1)	0	0	>	0	>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,13}$	L(3, 1)⊕L(1, 1)		>	0	0	>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,14}$	L(3, 1)⊕L(1, 1)	0	>	0	0	>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,15}$	L(3, 1)⊕L(1, 1)		0	0	0	>	$b_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,16}$	L(3, 1)⊕L(1, 1)		>	>		0	$a_3=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,17}$	L(3, 1)⊕L(1, 1)	0	>	>		0	$a_3=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,18}$	L(3, 1)⊕L(1, 1)		0	>		0	$a_3=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,19}$	L(3, 1)⊕L(1, 1)	0	0	>		0	$a_3=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,20}$	L(3, 1)⊕L(1, 1)		>	0	$\neq a_1$	0	$a_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,21}$	L(3, 1)⊕L(1, 1)		>	0	a_1	0	$a_2=1$	$\tilde{g}_{8,7}$	$\tilde{g}^d_{8,7}$
$\tilde{g}''_{4,22}$	L(3, 1)⊕L(1, 1)	0	>	0		0	$a_2=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}'_{4,1}$	4L(1, 1)		0	0	$\neq a_1$	0	$a_1=\pm 1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}'_{4,2}$	4L(1, 1)	(>)	0	0	a_1	0	$a_1=\pm 1$	$\tilde{g}_{8,7}$	$\tilde{g}^d_{8,7}$
$\tilde{g}''_{4,23}$	L(3, 1)⊕L(1, 1)		>	>	0	0	$a_3=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,24}$	L(3, 1)⊕L(1, 1)	0	>	>	0	0	$a_3=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,25}$	L(3, 1)⊕L(1, 1)		0	>	0	0	$a_3=1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}''_{4,26}$	L(3, 1)⊕L(1, 1)	0	>	0	0	0	$a_2=1$	$\tilde{g}_{8,7}$	$\tilde{g}^d_{8,7}$
$\tilde{g}'_{4,3}$	4L(1, 1)	(>)	0	0	0	0	$a_1=\pm 1$	$\tilde{g}_{7,10}$	$\tilde{g}^d_{7,10}$
$\tilde{g}'_{4,4}$	4L(1, 1)	0	0	0	0	0		$\tilde{g}_{10,1}$	$\tilde{g}^d_{11,2}$

$\tilde{G}^d$ conjugacy classes of such algebras coincides with the list given in table 3, where the range of parameters is given in the last two subcolumns of column 6. We shall denote these subalgebras $\tilde{g}^d_{i,k}\equiv\tilde{g}_{i,k}$, with $\tilde{g}_{i,k}$ given in table 3.

(2) Subalgebras of $\tilde{g}^d$ obtained from $f_1=\{d\}$, which are splitting extensions of subalgebras of $\tilde{g}$. They are obtained by adding the element d to a subalgebra $\tilde{g}_{i,k}$ of $\tilde{g}$ (classified under $\tilde{G}_0^d$, or $\tilde{G}^d$), which is an invariant subspace of d. We thus obtain algebras of the form $d\dotplus\tilde{g}_{i,k}$, where $\tilde{g}_{i,k}$ is one of the subalgebras listed in table 3. In these bases given in table 3 the d-invariant subalgebras $\tilde{g}_{i,k}$ are those which either involve no parameters (when classified under $\tilde{G}_0$), or contain parameters relating the translations p_i only. Thus, e.g., algebra $\tilde{g}_{4,14}$ is allowed, but $\tilde{g}_{4,15}$ is forbidden, as are all subalgebras with basis elements of the type $j_3+ak_i+bp_j$, k_3+at, $t+am$, k_i+bp_j, etc.

(3) Subalgebras of $\tilde{g}^d$ obtained from $f_1=\{d\}$, that are non-splitting extensions of subalgebras of $\tilde{g}$. They are obtained in the following manner.

(i) Take a subalgebra $\tilde{g}_{i,k}$ from table 3 that is an invariant subspace of d (i.e. the same subalgebras that were used above in case 2).

(ii) Add to $\tilde{g}_{i,k}$ a basis element of the form

$$d+a_ij_i+b_ik_i+c_ip_i+et+fm\equiv d+n \qquad a_i, b_i, c_i, e, f\in\mathbb{R} \tag{3.7}$$

such that

$$n \in [\mathrm{nor}_{\tilde{g}}\, \tilde{g}_{i,k}]/\tilde{g}_{i,k}. \tag{3.8}$$

The condition (3.8) ensures that $(d+n)\dot{+}\tilde{g}_{i,k}$ forms a Lie algebra and that $d+n$ cannot be simplified by linear combinations with elements of $\tilde{g}_{i,k}$.

(iii) Classify the elements (3.7) into conjugacy classes under the action of the normaliser of $\tilde{g}_{i,k}$ in the group $\tilde{G}_0^d$ (or $\tilde{G}^d$). Choose a representative of each conjugacy class for which at least one of the coefficients $a_i, b_i, \ldots, f$ is non-zero (if all coefficients are zero, up to conjugacy, we re-obtain a splitting subalgebra). The results of this procedure can be stated quite simply: the basis element (3.7) must actually have the form

$$\tilde{d} = d + aj_3 + bm \qquad (a, b) \neq (0,0) \qquad a, b \in \mathbb{R}. \tag{3.9}$$

Indeed, terms of the form $b_ik_i + c_ip_i + et$ can be removed by transformations in the Lie group generated by $\{\boldsymbol{k}, \boldsymbol{p}, m\}$. Terms of the form a_ij_i can be rotated into aj_3.

In table 5 we give a list of representatives of conjugacy classes of subalgebras $\tilde{g}^d_{i,k}$ of the extended Galilei-similitude algebra $\tilde{g}^d$, that involve a dilation. A complete list of representatives of all conjugacy classes of subalgebras of $\tilde{g}^d$ is obtained by merging tables 3 and 5 together. In the tables, e(3) denotes the Euclidean Lie algebra and sim(3) the similitude algebra of Euclidean 3-space (sim(3) $\sim \{d, \boldsymbol{j}, \boldsymbol{p}\}$).

Table 5. Representative of $\tilde{G}_0^d$ and $\tilde{G}^d$ conjugacy classes of subalgebras of the extended Galilei-similitude algebra $\tilde{g}^d$ involving a dilation.

Number	Isomorphism class and comments	Basis	$\mathrm{nor}_{\tilde{g}^d}$	Range of parameters $\tilde{G}_0^d$	Range of parameters $\tilde{G}^d$
$\tilde{g}^d_{12,1}$	$\tilde{g}^d(3)$	$d; \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{p}, t, m$	$\tilde{g}^d_{12,1}$		
$\tilde{g}^d_{11,2}$		$d; \boldsymbol{j}, \boldsymbol{k}, \boldsymbol{p}, m$	$\tilde{g}^d_{11,2}$		
$\tilde{g}^d_{10,2}$		$d, j_3; \boldsymbol{k}, \boldsymbol{p}, t, m$	$\tilde{g}^d_{10,2}$		
$\tilde{g}^d_{9,2}$	$\tilde{g}^d_{8,19}\oplus L(1,1)$	$\{d; \boldsymbol{j}, \boldsymbol{p}, t\}\oplus\{m;\}$	$\tilde{g}^d_{9,2}$		
$\tilde{g}^d_{9,3}$	solv, NR $=\tilde{g}^d_{8,3}$	$d; t, \boldsymbol{k}, \boldsymbol{p}, m$	$\tilde{g}^d_{12,1}$		
$\tilde{g}^d_{9,4}$	solv, NR $=\tilde{g}^d_{8,3}$	$d+aj_3; t, \boldsymbol{k}, \boldsymbol{p}, m$	$\tilde{g}^d_{10,2}$	$a>0$	$a>0$
$\tilde{g}^d_{9,5}$	solv, NR $=L(6,14;1)\oplus L(1,1)$	$d, j_3; k_1, k_2, \boldsymbol{p}, t, m$	$\tilde{g}^d_{9,5}$		
$\tilde{g}^d_{9,6}$	solv, NR $=h(3)$	$d, j_3; \boldsymbol{k}, \boldsymbol{p}, m$	$\tilde{g}^d_{9,6}$		
$\tilde{g}^d_{8,9}$	sim(3)$\oplus L(1,1)$	$\{d; \boldsymbol{j}, \boldsymbol{k}\}\oplus\{m;\}$	$\tilde{g}^d_{8,9}$		
$\tilde{g}^d_{8,10}$	sim(3)$\oplus L(1,1)$	$\{d; \boldsymbol{j}, \boldsymbol{p}\}\oplus\{m;\}$	$\tilde{g}^d_{8,10}$		
$\tilde{g}^d_{8,11}$	solv, NR $=L(6,14;1)\oplus L(1,1)$	$d; t, k_1, k_2, \boldsymbol{p}, m$	$\tilde{g}^d_{9,5}$		
$\tilde{g}^d_{8,12}$	solv, NR $=L(6,14;1)\oplus L(1,1)$	$d+aj_3; t, k_1, k_2, \boldsymbol{p}, m$	$\tilde{g}^d_{9,5}$	$a>0$	$a>0$
$\tilde{g}^d_{8,13}$	solv, NR $=h(3)$	$d; \boldsymbol{k}, \boldsymbol{p}, m$	$\tilde{g}^d_{11,2}$		
$\tilde{g}^d_{8,14}$	solv, NR $=h(3)$	$d+aj_3; \boldsymbol{k}, \boldsymbol{p}, m$	$\tilde{g}^d_{9,6}$	$a>0$	$a>0$
$\tilde{g}^d_{8,15}$	solv, NR $=L(4,1)\oplus 2L(1,1)$	$d, j_3; t, k_3, \boldsymbol{p}, m$	$\tilde{g}^d_{8,15}$		
$\tilde{g}^d_{8,16}$	solv, NR $=L(5,4)\oplus L(1,1)$	$d, j_3; \boldsymbol{k}, p_1, p_2, m$	$\tilde{g}^d_{8,16}$		
$\tilde{g}^d_{8,17}$	$\sim\tilde{g}^d_{8,16}$	$d, j_3; k_1, k_2, \boldsymbol{p}, m$	$\tilde{g}^d_{8,17}$		
$\tilde{g}^d_{8,18}$	solv, NR $=L(6,14;1)$	$d, j_3; t, k_1, k_2, p_1, p_2, m$	$\tilde{g}^d_{9,5}$		
$\tilde{g}^d_{8,19}$	non-solv	$d; \boldsymbol{j}, \boldsymbol{p}, t$	$\tilde{g}^d_{9,2}$		
$\tilde{g}^d_{8,20}$	$\sim\tilde{g}^d_{8,19}$	$d+am; \boldsymbol{j}, \boldsymbol{p}, t$	$\tilde{g}^d_{9,2}$	$a>0$	$a>0$
$\tilde{g}^d_{7,19}$	$\tilde{g}^d_{6,39}\oplus L(1,1)$	$\{d, j_3; t, \boldsymbol{p}\}\oplus\{m;\}$	$\tilde{g}^d_{7,19}$		
$\tilde{g}^d_{7,20}$	solv, NR $=L(4,1)\oplus 2L(1,1)$	$d; t, k_3, \boldsymbol{p}, m$	$\tilde{g}^d_{8,15}$		
$\tilde{g}^d_{7,21}$	solv, NR $=L(4,1)\oplus 2L(1,1)$	$d+aj_3; t, k_3, \boldsymbol{p}, m$	$\tilde{g}^d_{8,15}$	$a>0$	$a>0$
$\tilde{g}^d_{7,22}$	solv, NR $=L(5,4)\oplus L(1,1)$	$d; k_1, k_2, \boldsymbol{p}, m$	$\tilde{g}^d_{8,17}$		
$\tilde{g}^d_{7,23}$	$\sim\tilde{g}^d_{7,22}$	$d; p_1, p_2, \boldsymbol{k}, m$	$\tilde{g}^d_{8,16}$		
$\tilde{g}^d_{7,24}$	solv, NR $=L(5,4)\oplus L(1,1)$	$d+aj_3, k_1, k_2, \boldsymbol{p}, m$	$\tilde{g}^d_{8,17}$	$a>0$	$a>0$
$\tilde{g}^d_{7,25}$	$\sim\tilde{g}^d_{7,24}$	$d+aj_3; \boldsymbol{k}, p_1, p_2, m$	$\tilde{g}^d_{8,16}$	$a>0$	$a>0$

Table 5. (continued)

Number	Isomorphism class and comments	Basis	nor_{g^d}	Range of parameters $\tilde{G}_0^d$	Range of parameters $\tilde{G}^d$
$\tilde{g}^d_{7,26}$	solv, NR = L(6, 14; 1)	$d;\, t, k_1, k_2, p_1, p_2, m$	$\tilde{g}^d_{8,18}$		
$\tilde{g}^d_{7,27}$	solv, NR = L(6, 14; 1)	$d+aj_3;\, t, k_1, k_2, p_1, p_2, m$	$\tilde{g}^d_{8,18}$	$a>0$	$a>0$
$\tilde{g}^d_{7,28}$	solv, NR = L(3, 1)⊕2L(1, 1)	$d, j_3;\, k, p_3, m$	$\tilde{g}^d_{7,28}$		
$\tilde{g}^d_{7,29}$	$\sim\tilde{g}_{7,28}$	$d, j_3;\, p, k_3, m$	$\tilde{g}^d_{7,29}$		
$\tilde{g}^d_{7,30}$	solv, NR = L(5, 4)	$d, j_3;\, k_1, k_2, p_1, p_2, m$	$\tilde{g}^d_{7,30}$		
$\tilde{g}^d_{7,31}$	non-solv	$d+am;\, j, k$	$\tilde{g}^d_{8,9}$	$a\in\mathbb{R}$	$a\geqslant 0$
$\tilde{g}^d_{7,32}$	$\sim\tilde{g}^d_{7,31}$	$d+am;\, j, p$	$\tilde{g}^d_{8,10}$	$a\in\mathbb{R}$	$a\geqslant 0$
$\tilde{g}^d_{6,25}$	L(3, 6)⊕L(2, 1)⊕L(1, 1)	$\{;j\}\oplus\{d;\,t\}\oplus\{m;\}$	$\tilde{g}^d_{6,25}$		
$\tilde{g}^d_{6,26}$	L(5, 48; 1; 0)⊕L(1, 1)	$\{j_3, d;\, k\}\oplus\{m;\}$	$\tilde{g}^d_{6,26}$		
$\tilde{g}^d_{6,27}$	L(5, 48; 1; 0)⊕L(1, 1)	$\{j_3, d;\, p\}\oplus\{m;\}$	$\tilde{g}^d_{6,27}$		
$\tilde{g}^d_{6,28}$	L(5, 48; −1; 0)⊕L(1, 1)	$\{j_3, d;\, k_1, k_2, p_3\}\oplus\{m;\}$	$\tilde{g}^d_{6,28}$		
$\tilde{g}^d_{6,29}$	L(5, 48; −1; 0)⊕L(1, 1)	$\{j_3, d;\, p_1, p_2, k_3\}\oplus\{m;\}$	$\tilde{g}^d_{6,29}$		
$\tilde{g}^d_{6,30}$	L(5, 7; 1/2; 1/2; 1/2)⊕L(1, 1)	$\{d;\, p, t\}\oplus\{m;\}$	$\tilde{g}^d_{9,2}$		
$\tilde{g}^d_{6,31}$	L(5, 48; 2; 0)⊕L(1, 1)	$\{j_3, d;\, p_1, p_2, t\}\oplus\{m;\}$	$\tilde{g}^d_{6,31}$		
$\tilde{g}^d_{6,32}$	L(5, 40)⊕L(1, 1)	$\{d;\, k_3, p_3, t, m\}\oplus\{j_3;\}$	$\tilde{g}^d_{6,32}$		
$\tilde{g}^d_{6,33}$	L(5, 10; 1/2, a/2, 1/2)⊕L(1, 1)	$\{d+aj_3;\, p, t\}\oplus\{m;\}$	$\tilde{g}^d_{7,19}$	$a>0$	$a>0$
$\tilde{g}^d_{6,34}$	solv, NR = L(3, 1)⊕2L(1, 1)	$d;\, k, p_3, m$	$\tilde{g}^d_{7,28}$		
$\tilde{g}^d_{6,35}$	$\sim\tilde{g}^d_{6,34}$	$d;\, p, k_3, m$	$\tilde{g}^d_{7,29}$		
$\tilde{g}^d_{6,36}$	$\sim\tilde{g}^d_{6,34}$	$d;\, k_1, k_2, p_1, p_3, m$	$\tilde{g}^d_{6,36}$		
$\tilde{g}^d_{6,37}$	solv, NR = L(4, 1)⊕L(1, 1)	$d;\, k_2, p_1, p_2, t, m$	$\tilde{g}^d_{6,37}$		
$\tilde{g}^d_{6,38}$	solv, NR = L(5, 4)	$d;\, k_1, k_2, p_1, p_2, m$	$\tilde{g}^d_{7,30}$		
$\tilde{g}^d_{6,39}$	$\sim\tilde{g}^d_{6,38}$	$d;\, k_1, k_2, p_2, p_1+ap_3, m$	$\tilde{g}^d_{6,39}$	$a>0$	$a>0$
$\tilde{g}^d_{6,40}$	solv, NR = 4L(1, 1)	$d, j_3;\, p, t$	$\tilde{g}^d_{7,19}$		
$\tilde{g}^d_{6,41}$	$\sim\tilde{g}^d_{6,40}$	$d+bm, j_3+am;\, p, t$	$\tilde{g}^d_{7,19}$	$a\geqslant 0$	$a\geqslant 0$
				$b\neq 0$	$b>0$
$\tilde{g}^d_{6,42}$	$\sim\tilde{g}^d_{6,40}$	$d, j_3+am;\, p, t$	$\tilde{g}^d_{7,19}$	$a>0$	$a>0$
$\tilde{g}^d_{6,43}$	solv, NR = L(3, 1)⊕2L(1, 1)	$d+aj_3;\, k, p_3, m$	$\tilde{g}^d_{7,28}$	$a>0$	$a>0$
$\tilde{g}^d_{6,44}$	$\sim\tilde{g}^d_{6,43}$	$d+aj_3;\, p, k_3, m$	$\tilde{g}^d_{7,29}$	$a>0$	$a>0$
$\tilde{g}^d_{6,45}$	solv, NR = L(5, 4)	$d+aj_3;\, k_1, k_2, p_1, p_2, m$	$\tilde{g}^d_{7,30}$	$a>0$	$a>0$
$\tilde{g}^d_{5,43}$	L(3, 2; 1/2)⊕2L(1, 1)	$\{d;\, p_3, t\}\oplus\{j_3;\}\oplus\{m;\}$	$\tilde{g}^d_{5,43}$		
$\tilde{g}^d_{5,44}$	L(3, 6)⊕2L(1, 1)	$\{;j\}\oplus\{m;\}\oplus\{d;\}$	$\tilde{g}^d_{5,44}$		
$\tilde{g}^d_{5,45}$	L(3, 6)⊕L(2, 1)	$\{;j\}\oplus\{d+am;\, t\}$	$\tilde{g}^d_{6,25}$	$a\in\mathbb{R}$	$a\geqslant 0$
$\tilde{g}^d_{5,46}$	L(4, 2; 1; 1)⊕L(1, 1)	$\{d;\, k\}\oplus\{m;\}$	$\tilde{g}^d_{8,9}$		
$\tilde{g}^d_{5,47}$	L(4, 2; 1; 1)⊕L(1, 1)	$\{d;\, p\}\oplus\{m;\}$	$\tilde{g}^d_{8,10}$		
$\tilde{g}^d_{5,48}$	L(4, 2; 1; −1)⊕L(1, 1)	$\{d;\, k_1, k_2, p_3\}\oplus\{m;\}$	$\tilde{g}^d_{6,28}$		
$\tilde{g}^d_{5,49}$	L(4, 2; 1; −1)⊕L(1, 1)	$\{d;\, p_1, p_2, k_3\}\oplus\{m;\}$	$\tilde{g}^d_{6,29}$		
$\tilde{g}^d_{5,50}$	L(4, 2; 1/2; 1/2)⊕L(1, 1)	$\{d;\, p_1, p_2, t\}\oplus\{m;\}$	$\tilde{g}^d_{6,31}$		
$\tilde{g}^d_{5,51}$	L(4, 5; 1/a; 1/a)⊕L(1, 1)	$\{d+aj_3;\, k\}\oplus\{m;\}$	$\tilde{g}^d_{6,26}$	$a>0$	$a>0$
$\tilde{g}^d_{5,52}$	L(4, 5; 1/a; 1/a)⊕L(1, 1)	$\{d+aj_3;\, p\}\oplus\{m;\}$	$\tilde{g}^d_{6,27}$	$a>0$	$a>0$
$\tilde{g}^d_{5,53}$	L(4, 5; 1/a; −1/a)⊕L(1, 1)	$\{d+aj_3;\, k_1, k_2, p_3\}\oplus\{m;\}$	$\tilde{g}^d_{6,28}$	$a>0$	$a>0$
$\tilde{g}^d_{5,54}$	L(4, 5; 1/a; −1/a)⊕L(1, 1)	$\{d+aj_3;\, p_1, p_2, k_3\}\oplus\{m;\}$	$\tilde{g}^d_{6,29}$	$a>0$	$a>0$
$\tilde{g}^d_{5,55}$	L(4, 5; 2/a; 1/a)⊕L(1, 1)	$\{d+aj_3;\, p_1, p_2, t\}\oplus\{m;\}$	$\tilde{g}^d_{6,31}$	$a>0$	$a>0$
$\tilde{g}^d_{5,56}$	L(4, 8)⊕L(1, 1)	$\{d;\, k_3, p_3, m\}\oplus\{j_3;\}$	$\tilde{g}^d_{5,56}$		
$\tilde{g}^d_{5,57}$	L(4, 13)⊕L(1, 1)	$\{j_3, d;\, k_1, k_2\}\oplus\{m;\}$	$\tilde{g}^d_{5,57}$		
$\tilde{g}^d_{5,58}$	L(4, 13)⊕L(1, 1)	$\{j_3, d;\, p_1, p_2\}\oplus\{m;\}$	$\tilde{g}^d_{5,58}$		
$\tilde{g}^d_{5,59}$	L(5, 7; 1/2; 1/2; 1/2)	$d+am;\, p, t$	$\tilde{g}^d_{9,2}$	$a\in\mathbb{R}$	$a\geqslant 0$
$\tilde{g}^d_{5,60}$	L(5,10; 1/2; a/2; 1/2)	$d+aj_3+bm;\, p, t$	$\tilde{g}^d_{7,19}$	$a>0$	$a>0$
				$b\in\mathbb{R}$	$b\geqslant 0$
$\tilde{g}^d_{5,61}$	L(5, 21; 1)	$d;\, k_1, k_2, p_1+ap_3, m$	$\tilde{g}^d_{5,61}$	$a\geqslant 0$	$a\geqslant 0$
$\tilde{g}^d_{5,62}$	L(5, 21; 1)	$d;\, k_3, p_1, p_3+ap_2, m$	$\tilde{g}^d_{5,62}$	$a\geqslant 0$	$a\geqslant 0$
$\tilde{g}^d_{5,63}$	L(5, 40)	$d+aj_3, k_3, p_3, t, m$	$\tilde{g}^d_{6,32}$	$a>0$	$a>0$
$\tilde{g}^d_{5,64}$	L(5, 40)	$d;\, k_3, p_3, t, m$	$\tilde{g}^d_{6,32}$		
$\tilde{g}^d_{5,65}$	L(5, 48; 1; 0)	$j_3+am, d+bm;\, k$	$\tilde{g}^d_{6,26}$	$a\geqslant 0$	$a\geqslant 0$
				$b\in\mathbb{R}$	$b\geqslant 0$
$\tilde{g}^d_{5,66}$	L(5, 48; 1; 0)	$j_3+am, d+bm;\, p$	$\tilde{g}^d_{6,27}$	$a\geqslant 0$	$a\geqslant 0$
				$b\in\mathbb{R}$	$b\geqslant 0$
$\tilde{g}^d_{5,67}$	L(5, 48; −1, 0)	$j_3+am, d+bm;\, k_3, p_1, p_2$	$\tilde{g}^d_{6,29}$	$a\geqslant 0$	$a\geqslant 0$
				$b\in\mathbb{R}$	$b\geqslant 0$

Table 5. (continued)

Number	Isomorphism class and comments	Basis	nor_{g^d}	Range of parameters $\tilde G_0^d$	$\tilde G^d$
$\tilde g^d_{5,68}$	L(5, 48; −1; 0)	$j_3+am, d+bm; p_3, k_1, k_2$	$\tilde g^d_{6,28}$	$a \geqslant 0$	$a \geqslant 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{5,69}$	L(5, 48; 2; 0)	$j_3+am, d+bm; p_1, p_2, t$	$\tilde g^d_{6,31}$	$a \geqslant 0$	$a \geqslant 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{4,56}$	L(2, 1)⊕2L(1, 1)	$\{d; k_3\}\oplus\{j_3;\}\oplus\{m;\}$	$\tilde g^d_{4,56}$		
$\tilde g^d_{4,57}$	L(2, 1)⊕2L(1, 1)	$\{d; t\}\oplus\{j_3;\}\oplus\{m;\}$	$\tilde g^d_{4,57}$		
$\tilde g^d_{4,58}$	L(2, 1)⊕2L(1, 1)	$\{d; p_3\}\oplus\{j_3;\}\oplus\{m;\}$	$\tilde g^d_{4,58}$		
$\tilde g^d_{4,59}$	L(3, 2; 1)⊕L(1, 1)	$\{d; k_1, k_2\}\oplus\{m;\}$	$\tilde g^d_{5,59}$		
$\tilde g^d_{4,60}$	L(3, 2; 1)⊕L(1, 1)	$\{d; p_1, p_2\}\oplus\{m;\}$	$\tilde g^d_{5,60}$		
$\tilde g^d_{4,61}$	L(3, 2; −1)⊕L(1, 1)	$\{d; k_3, p_1\}\oplus\{m;\}$	$\tilde g^d_{4,61}$		
$\tilde g^d_{4,62}$	L(3, 2; 1/2)⊕L(1, 1)	$\{d+aj_3; t, p_3\}\oplus\{m;\}$	$\tilde g^d_{5,43}$	$a \geqslant 0$	$a \geqslant 0$
$\tilde g^d_{4,63}$	L(3, 2; 1/2)⊕L(1, 1)	$\{d+bm; t, p_3\}\oplus\{j_3+am;\}$	$\tilde g^d_{5,43}$	$a \geqslant 0$	$a \geqslant 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{4,64}$	L(3, 4; 1/a)⊕L(1, 1)	$\{d+aj_3; k_1, k_2\}\oplus\{m;\}$	$\tilde g^d_{5,57}$	$a > 0$	$a > 0$
$\tilde g^d_{4,65}$	L(3, 4; 1/a)⊕L(1, 1)	$\{d+aj_3; p_1, p_2\}\oplus\{m;\}$	$\tilde g^d_{5,58}$	$a > 0$	$a > 0$
$\tilde g^d_{4,66}$	L(3, 6)⊕L(1, 1)	$\{; \mathbf{j}\}\oplus\{d+am;\}$	$\tilde g^d_{5,44}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde g^d_{4,67}$	L(4, 2; 1; 1)	$d+am; \mathbf{k}$	$\tilde g^d_{8,9}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde g^d_{4,68}$	L(4, 2; 1; 1)	$d+am; \mathbf{p}$	$\tilde g^d_{8,10}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde g^d_{4,69}$	L(4, 2; 1; −1)	$d+am; k_1, k_2, p_3$	$\tilde g^d_{6,28}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde g^d_{4,70}$	L(4, 2; 1; −1)	$d+am; p_1, p_2, k_3$	$\tilde g^d_{6,29}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde g^d_{4,71}$	L(4, 2; 1/2; 1/2)	$d+am; p_1, p_2, t$	$\tilde g^d_{6,31}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde g^d_{4,72}$	L(4, 5; 1/a; 1/a)	$d+aj_3+bm; \mathbf{k}$	$\tilde g^d_{6,26}$	$a > 0$	$a > 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{4,73}$	L(4, 5; 1/a; 1/a)	$d+aj_3+bm; \mathbf{p}$	$\tilde g^d_{6,27}$	$a > 0$	$a > 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{4,74}$	L(4, 5; 1/a; −1/a)	$d+aj_3+bm; k_1, k_2, p_3$	$\tilde g^d_{6,28}$	$a > 0$	$a > 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{4,75}$	L(4, 5; 1/a; −1/a)	$d+aj_3+bm; p_1, p_2, k_3$	$\tilde g^d_{6,29}$	$a > 0$	$a > 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{4,76}$	L(4, 5; 2/a; 1/a)	$d+aj_3+bm; p_1, p_2, t$	$\tilde g^d_{6,31}$	$a > 0$	$a > 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{4,77}$	L(4, 8)	$d; k_3, p_3, m$	$\tilde g^d_{5,56}$		
$\tilde g^d_{4,78}$	L(4, 8)	$d; k_3, p_3+ap_1, m$	$\tilde g^d_{4,77}$	$a > 0$	$a > 0$
$\tilde g^d_{4,79}$	L(4, 8)	$d+aj_3; k_3, p_3, m$	$\tilde g^d_{5,56}$	$a > 0$	$a > 0$
$\tilde g^d_{4,80}$	L(4, 13)	$d+bm, j_3+am; k_1, k_2$	$\tilde g^d_{5,57}$	$a \geqslant 0$	$a \geqslant 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{4,81}$	L(4, 13)	$d+bm, j_3+am; p_1, p_2$	$\tilde g^d_{5,58}$	$a \geqslant 0$	$a \geqslant 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{3,53}$	3L(1, 1)	$\{d;\}\oplus\{j_3;\}\oplus\{m;\}$	$\tilde g^d_{3,53}$		
$\tilde g^d_{3,54}$	L(2, 1)⊕L(1, 1)	$\{d+bm; k_3\}\oplus\{j_3+am;\}$	$\tilde g^d_{4,56}$	$a \geqslant 0$	$a \geqslant 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{3,55}$	L(2, 1)⊕L(1, 1)	$\{d+bm; t\}\oplus\{j_3+am;\}$	$\tilde g^d_{4,57}$	$a \geqslant 0$	$a \geqslant 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{3,56}$	L(2, 1)⊕L(1, 1)	$\{d+bm; p_3\}\oplus\{j_3+am;\}$	$\tilde g^d_{4,58}$	$a \geqslant 0$	$a \geqslant 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{3,57}$	L(2, 1)⊕L(1, 1)	$\{d+aj_3; k_3\}\oplus\{m;\}$	$\tilde g^d_{4,56}$	$a \geqslant 0$	$a \geqslant 0$
$\tilde g^d_{3,58}$	L(2, 1)⊕L(1, 1)	$\{d+aj_3; t\}\oplus\{m;\}$	$\tilde g^d_{4,57}$	$a \geqslant 0$	$a \geqslant 0$
$\tilde g^d_{3,59}$	L(2, 1)⊕L(1, 1)	$\{d+aj_3; p_3\}\oplus\{m;\}$	$\tilde g^d_{4,58}$	$a \geqslant 0$	$a \geqslant 0$
$\tilde g^d_{3,60}$	L(3, 2; 1)	$d+am; k_1, k_2$	$\tilde g^d_{5,57}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde g^d_{3,61}$	L(3, 2; 1)	$d+am; p_1, p_2$	$\tilde g^d_{5,58}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde g^d_{3,62}$	L(3, 2; −1)	$d+am; k_3, p_1$	$\tilde g^d_{4,61}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde g^d_{3,63}$	L(3, 2; 1/2)	$d+aj_3+bm; t, p_3$	$\tilde g^d_{5,43}$	$a > 0$	$a > 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{3,64}$	L(3, 2; 1/2)	$d+am; t, p_3$	$\tilde g^d_{5,43}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde g^d_{3,65}$	L(3, 4; 1/a)	$d+aj_3+bm; k_1, k_2$	$\tilde g^d_{5,57}$	$a > 0$	$a > 0$
				$b \in \mathbb{R}$	$b \geqslant 0$
$\tilde g^d_{3,66}$	L(3, 4; 1/a)	$d+aj_3+bm; p_1, p_2$	$\tilde g^d_{5,58}$	$a > 0$	$a > 0$
				$b \in \mathbb{R}$	$b \geqslant 0$

Table 5. (continued)

Number	Isomorphism class and comments	Basis	$\mathrm{nor}_{\mathrm{g}^{\mathrm{d}}}$	Range of parameters $\tilde{G}_0^d$	$\tilde{G}^d$
$\tilde{g}^d_{2,28}$	2L(1, 1)	$\{j_3 + am;\} \oplus \{d + bm;\}$	$\tilde{g}^d_{3,53}$	$a \geqslant 0$, $b \in \mathbb{R}$	$a \geqslant 0$, $b \geqslant 0$
$\tilde{g}^d_{2,29}$	2L(1, 1)	$\{d + aj_3;\} \oplus \{m;\}$	$\tilde{g}^d_{3,53}$	$a > 0$	$a > 0$
$\tilde{g}^d_{2,30}$	2L(1, 1)	$\{d;\} \oplus \{m;\}$	$\tilde{g}^d_{5,44}$		
$\tilde{g}^d_{2,31}$	L(2, 1)	$d + aj_3 + bm;\ k_3$	$\tilde{g}^d_{4,56}$	$a > 0$, $b \in \mathbb{R}$	$a > 0$, $b \geqslant 0$
$\tilde{g}^d_{2,32}$	L(2, 1)	$d + am;\ k_3$	$\tilde{g}^d_{4,56}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde{g}^d_{2,33}$	L(2, 1)	$d + aj_3 + bm;\ t$	$\tilde{g}^d_{4,55}$	$a \geqslant 0$, $b \in \mathbb{R}$	$a > 0$, $b \geqslant 0$
$\tilde{g}^d_{2,34}$	L(2, 1)	$d + am;\ t$	$\tilde{g}^d_{6,25}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde{g}^d_{2,35}$	L(2, 1)	$d + aj_3 + bm;\ p_3$	$\tilde{g}^d_{4,58}$	$a > 0$, $b \in \mathbb{R}$	$a > 0$, $b \geqslant 0$
$\tilde{g}^d_{2,36}$	L(2, 1)	$d + am;\ p_3$	$\tilde{g}^d_{4,58}$	$a \in \mathbb{R}$	$a \geqslant 0$
$\tilde{g}^d_{1,13}$	L(1, 1)	$d + aj_3 + bm;$	$\tilde{g}^d_{3,53}$	$a > 0$, $b \in \mathbb{R}$	$a > 0$, $b \geqslant 0$
$\tilde{g}^d_{1,14}$	L(1, 1)	$d + am;$	$\tilde{g}^d_{5,43}$	$a \in \mathbb{R}$	$a \geqslant 0$

4. Conclusions

The results of this article can be summarised as follows. Any subalgebra of the extended Galilei algebra $\tilde{g}(3)$ is conjugated under the group of inner automorphisms to precisely one subalgebra in the list of table 3. Any subalgebra of the extended Galilei-similitude algebra is conjugated to precisely one subalgebra in the lists of table 3, or table 5.

The immediate application of this subgroup classification will be to obtain group invariant solutions of the GNLSE (1.1). A different application concerns the question of symmetry breaking for equation (1.1). Thus, for each subgroup, we can find the most general second-order equation, invariant under this subgroup. In general, there will be equations that describe more general interactions than (1.1) but reduce to (1.1) when these interaction terms are set equal to zero.

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References

[1] Hasegawa A and Kodama Y 1982 *Opt. Lett.* 7 285; 1981 *Proc. IEEE* **69** 1145
[2] Turner J G and Boyd J J M 1979 *J. Plasma Phys.* **22** 121
[3] Martina L 1986 *Preprint* Montréal CRM-1408
[4] Landau L D and Lifshitz E M 1980 *Statistical Physics* (Oxford: Pergamon)

[5] Davydov A S 1984 *Solitons in Molecular Systems* (Kiev: Naukova Dumka) (in Russian)
Paul P, Chatterjee R, Tuszynski J A and Fritz O G 1983 *J. Theor. Biol.* **104** 169
[6] Pushkarov Kh I, Pushkarov D I and Tomov I V 1979 *Opt. Quantum Electron.* **11** 471
[7] Cowan S, Enns R H, Rangnekar S S and Sanghera S S 1986 *Can. J. Phys.* **64** 311
[8] Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia:SIAM)
[9] Calogero F and Degasperis A 1982 *Spectral Transform and Solitons* vol 1 (Amsterdam: North-Holland)
[10] Lie S 1891 *Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen,* (Leipzig: Teubner)
[11] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)
[12] Bluman G W and Cole J D 1974 *Similarity Methods for Differential Equations* (Berlin: Springer)
[13] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)
[14] Winternitz P 1983 *Nonlinear Phenomena* (*Lecture Notes in Physics* **189**) (Berlin: Springer) p 263
[15] Champagne B and Winternitz P 1985 *Preprint* Montréal CRM-1278
[16] Schwartz F 1985 *Computing* **34** 91
[17] Patera J, Winternitz P and Zassenhaus H 1975 *J. Math. Phys.* **16** 1597
[18] Patera J, Winternitz P and Zassenhaus H 1975 *J. Math. Phys.* **16** 1615
[19] Patera J, Winternitz P and Zassenhaus H 1976 *J. Math. Phys.* **17** 717
[20] Patera J, Sharp R T, Winternitz P and Zassenhaus H 1976 *J. Math. Phys.* **17** 977; 1977 *J. Math. Phys.* **18** 2259
[21] Boyer C P, Sharp R T and Winternitz P 1976 *J. Math. Phys.* **17** 1439
[22] Burdet G, Patera J, Perrin M and Winternitz P 1978 *Ann. Sci. Math. Qué.* **2** 81; 1978 *J. Math. Phys.* **19** 1758
[23] Ince E L 1956 *Ordinary Differential Equations* (New York: Dover)
[24] Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715
[25] Rand D and Winternitz P 1986 *Comput. Phys. Commun.* **42** 359
[26] Bureau F J 1972 *Ann. Mat. Pura Appl.* IV **41** 164
[27] Grundland A M, Harnad J and Winternitz P 1984 *J. Math. Phys.* **25** 791
[28] Winternitz P, Grundland A M and Tuszynski J 1987 *J. Math. Phys.* **28** 2194
[29] Grundland A M, Tuszynski J and Winternitz P 1987 *Phys. Lett.* **119A** 340
[30] David D, Kamran N, Levi D and Winternitz P 1985 *Phys. Rev. Lett.* **55** 2111; 1986 *J. Math. Phys.* **27** 1225
[31] David D, Levi D and Winternitz P 1986 *Phys. Lett.* **118A** 390
[32] Champagne B and Winternitz P 1985 *Preprint* Montréal CRM-1414
[33] Leo R A, Martina L and Soliani G 1986 *J. Math. Phys.* **27** 2623
[34] Dorizzi B, Grammaticos B, Ramani A and Winternitz P 1986 *J. Math. Phys.* **27** 2848
[35] Sorba P 1976 *J. Math. Phys.* **17** 941
[36] Jacobson N 1979 *Lie Algebras* (New York: Dover)
[37] Levy-Leblond J M 1974 *Group Theory and its Applications* vol 2, ed E Loebl (New York: Academic)
[38] Voisin J 1965 *J. Math. Phys.* **6** 1519, 1822
[39] Gagnon L 1987 *Preprint* Montréal CRM-1478
[40] Mubarakzyanov G M 1963 *Izv. Vyssh. Uchebn. Zaved. Mat.* no 1 114; no 3 99; no 4 104
[41] Morozov V V 1958 *Izv. Vyssh. Uchebn. Zaved. Mat.* no 4 161
[42] Patera J, Sharp R T, Winternitz P and Zassenhaus H 1976 *J. Math. Phys.* **17** 986
[43] Rand D, Winternitz P and Zassenhaus H 1988 *Linear Algebra Appl.* to be published
[44] Rand D, Winternitz P and Zassenhaus H 1988 *Linear Algebra Appl.* to be published